Math 215 - Final

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This exam is due Monday, December 18 by noon. Please attach this to the front of your work. By signing the bottom of this page you are acknowledging that you did not give or receive help on this exam. You can use your textbook, notes and information linked on the course webpage. You should complete this exam in one 4 hour sitting.

Throughout all rings will be commutative with identity.

1. Let A be an abelian group. Define a multiplication on $\mathbb{Z} \times A$ by $(m, a) \cdot (n, b) = (mn, mb + na)$
   (a) Show that this defines a ring structure on $\mathbb{Z} \times A$.
   (b) Find the group of units of this ring structure. Show that this is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times A$.
   (c) Find the zero divisors of this ring.
2. Let $R = \mathbb{Z}[x]$. Let I be the principal ideal generated by $x^2 - 3x + 2$. Show that R/I is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Hint: Consider the map $\phi \colon R \to \mathbb{Z} \times \mathbb{Z}$ given by $\phi(f) = (f(1), f(2))$.
3. Let R_1 and R_2 be commutative rings.
   (a) Show that an ideal I in $R_1 \times R_2$ is of the form $I_1 \times I_2$ for ideals $I_1 = \{a \in R_1 \mid (a, b) \in I\}$ and $I_2 = \{b \in R_2 \mid (a, b) \in I\}$.
   (b) Show that prime ideals are of the form $P_1 \times R_2$ or $R_1 \times P_2$, where P_i is an ideal of R_i.
4. (a) Let R be a PID. Show that if $P_1 \subsetneq P_2 \subsetneq R$ are prime ideals of R, then $P_1 = (0)$.
   (b) Give an example of a integral domain and prime ideals $P_1 \subsetneq P_2 \subsetneq R$ such that $P_1 \neq (0)$.
5. Given a module M over an integral domain R the *torsion submodule* is the set $\{m \in M \mid \exists r \in R \setminus \{0\}, rm = 0\}$. A module is *torsion free* is the torsion submodule is $\{0\}$.

   Let R be an integral domain. Let M be an R-module and N be a submodule.

(a) Show that if N and M/N are torsion free, then M is torsion free.

(b) Show that if R is a field then all modules are torsion free.

(c) Show that the converse of (a) is true if and only if R is a field.

6. Let $R = \mathbb{R}[x, y]$ and let $I = (x, y)$ be the ideal generated by x and y. Prove or disprove the following:

   (a) I is a free R-module.

   (b) I is a free $\mathbb{R}[x]$-module.

   (c) I is a free $\mathbb{R}$-module.

7. Find the following $\mathbb{Z}$-modules.

   (a) $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}$

   (b) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$

   (c) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.

8. Using the universal property for the tensor product and induction. Show that the iterated tensor product $M_1 \otimes_R \cdots \otimes_R M_k$ has the following universal property. For any R-module L and any map $K\colon M_1 \times \cdots \times M_k \to L$ which is linear in each component, there is an R-module homomorphism $\phi\colon M_1 \otimes_R \cdots \otimes_R M_k \to L$ such that $\phi(m_1 \otimes \cdots \otimes m_k) = K(m_1, \ldots, m_k)$.