Problem Sheet 4

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1. Let g, h be elements of a group G such that $gh = hg$ compute the order of gh. Let $\sigma \in S_n$, compute the order of σ. (Hint: Write σ in cycle notation first.)

2. We will show that for $n \neq 6$ the automorphism group $Aut(S_n) = Inn(S_n) \cong S_n$. Fix an integer $n > 1, n \neq 6$.

   (a) Prove that any automorphism permutes conjugacy classes. I.e. for each $\phi \in Aut(G)$ and each conjugacy class $\mathcal{K}$ the set $\phi(\mathcal{K})$ is also a conjugacy class of G.

   (b) (Extra Credit) Let $\mathcal{K}$ be the conjugacy class of the transpositions. Show that there are no other conjugacy classes of order 2 elements with the same size as $\mathcal{K}$. (Use the formula on Dummit and Foote pg. 132)

   For the rest of the question you may assume part b)

   (c) Prove that for each $\phi \in Aut(S_n)$

   $$\phi((1\,2)) = (a\,b_1), \phi((1\,3)) = (a\,b_2), \ldots, \phi((1\,n)) = (a\,b_n)$$

   (d) Show that $(1\,2), \ldots, (1\,n)$ generate S_n and deduce that any automorphism of S_n is determined by what it does to these elements.

   (e) Deduce that $Aut(S_n) cong S_n$ unless $n = 6$

3. Show that if $n_q(G) = 1 = n_p(G)$, P is the unique Sylow p-subgroup and Q is the unique Sylow q-subgroup, then every element of P commutes with every element of Q.

4. Show that any group of order 45 is abelian.

5. Show that a group of order 351 has a normal Sylow p-subgroup for some p dividing 351

6. Prove that if $|G| = 6545$, then G is not simple.

7. Prove that if $|G| = 2907$, then G is not simple.