Problem Sheet 5

Robert Kropholler

October 6, 2017

1. Show that if A and B are normal subgroups of G and G/A and G/B are abelian, then $G/A \cap B$ is abelian.

2. Let D_{2n} be the group of isometries of a regular n-gon S (maps $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(S) = S$ and distance is preserved). For more details look at chapter 1.2 in Dummit and Foote.

   (a) Show that D_{2n} has 2n elements. (Show that any element is determined by where it sends 2 adjacent vertices of the polygon. Show that any such choice extends to an isometry.)

   (b) Show that any element of D_{2n} is either a rotation or a rotation composed with a reflection.

   (c) Show that the subgroup of rotations is a normal subgroup N.

   (d) Show that there is a subgroup K of order 2 such that $N \cap K = \{e\}$.

   (e) By considering the action of K on H show that $D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z}$ and describe the map φ.

3. Let G be a finite group such that every element has order 2. Show that G is abelian. Moreover show that $G \cong \prod_{i=1}^{n} \mathbb{Z}/2\mathbb{Z}$ for some n.

4. Let $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $N = \mathbb{Z}/4\mathbb{Z}$. Let $K = \mathbb{Z}/2\mathbb{Z}$. Let K act on H by inverting elements, and K act on N by inverting elements (you do not need to prove that these are automorphism.) Show that $H \rtimes K$ is not isomorphic to $N \rtimes K$. (Consider the number of elements of order 2.)

5. There are only 2 homomorphisms from $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ and also only two homomorphisms $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/4\mathbb{Z})$. Use this and the above to show that there are 5 groups of order 8. (You should get 3 abelian groups and 2 non-abelian groups. You may also assume the classification of finitely generated abelian groups.)