Algebra 215 Problem Sheet 6

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Throughout R and S will be commutative rings and I will be an ideal of R. An element x of R is *nilpotent* if $x^m = 0$ for some $m \in \mathbb{N}$.

1. Show that any nilpotent element is either 0 or a zero divisor.
2. Find all the nilpotent elements in $\mathbb{Z}/60\mathbb{Z}$.
3. Show that the set of all zero divisors forms a prime ideal.
4. Let r be a unit in R and n be a nilpotent element. Show that $r + n$ is a unit.
5. Show that the set of nilpotent elements $\mathcal{N}(R)$ is an ideal of R. [You should use the binomial theorem to show that this is closed under addition]
6. Show that $\mathcal{N}(R/\mathcal{N}) = \{0\}$.
7. Let $\phi\colon R \to S$ be a ring homomorphism. Show that if x is nilpotent $\phi(x)$ is nilpotent.
8. Assume I is a prime ideal, show that $\mathcal{N}(R) \subset I$.
9. Let $\mathrm{rad}(I) = \{x \in R \mid \exists m \in \mathbb{N} \text{ s.t. } x^m \in I\}$. Show that $\mathrm{rad}(I)$ is an ideal of R containing I.
10. Show that $\mathcal{N}(R/I) = \mathrm{rad}(I)/I$.
11. Show that if I is a prime ideal $\mathrm{rad}(I) = I$.
12. Prove the following are equivalent:
    (a) R has one prime ideal.
    (b) $R/\mathcal{N}$ is a field.
    (c) Every element of R is either a unit or nilpotent.