Algebra Problem Sheet 7

Robert Kropholler

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1. Let R be a principal ideal domain. Let D be a multiplicatively closed subset of a ring R. Show that $D^{-1}R$ is a principal ideal domain.

2. Let R be an integral domain and suppose that every prime ideal is principal.

   (a) Assume the set of ideals of R that are not principal is nonempty and use Zorn's Lemma to show that there is a maximal element.

   (b) Let I be the maximal element from above, note this is not a prime ideal since it is not principal. Let $a, b \in R$ be elements of the ring R such that $ab \in I$ but $a \notin I$ and $b \notin I$. Let $I_a = (I, a)$ be the ideal generated by I and a, let I_b be defined similarly, and define $J = \{r \in R \mid rI_a \subset I\}$. Prove that $I_a = (\alpha)$ and $J = (\beta)$ are principal ideals in R with $I \subsetneq I_b \subset J$ and $I_aJ \subset I$.

   (c) If $x \in I$ show that $x = s\alpha$ for some $s \in J$. Deduce that $I = I_aJ$ and that I is principal. Use this contradiction to prove that R is a PID.

3. Read the section in Dummit and Foote on the "Factorisation in the Guassian integers" p.289 – 291

4. Let $R = \mathbb{Z}[i]$ be the Guassian integers.

   (a) Show that $R/(1 + i)$ is a field with 2 elements.

   (b) Let q be a prime equivalent to $1 \mod 4$. Show that $R/(q)$ is a field with q^2 elements.

5. Let R be a unique factorisation domain. The *least common multiple* of a and b, denoted $\mathrm{lcm}(a, b)$ is an element r such that $a|r$ and $b|r$ and if $a|s$ and $b|s$, then $r|s$.

   (a) Prove that $\mathrm{lcm}(a, b) = \gcd\{x \mid x = ra, x = sb\}$.

   (b) Show that the ideal $(a) \cap (b)$ is principal and generated by the least common multiple of a and b.

   (c) Given factorisations of a and b describe the least common multiple of a and b in terms of these factorisations.