Sheet 9 Solutions

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1. Let $A_1, \dots, A_n$ be R-modules. Let $B_i \subset A_i$ be submodules. Show that $(A_1 \oplus \dots \oplus A_n)/(B_1 \oplus \dots \oplus B_n)$ is isomorphic to $A_1/B_1 \oplus \dots A_n/B_n$.

   Proof. Define a homomorphism $\phi\colon A_1, \dots, A_n \to A_1/B_1 \oplus \dots A_n/B_n$, by naturally projecting on each factor. It is clear that this map is surjective. If we can show that the kernel of this map is $B = (B_1 \oplus \dots \oplus B_n)$ we will be done by the first isomorphism theorem.

   Any element of B maps to the identity since the projection to each factor maps to the identity. Further an element of the kernel is of the form $(x_1, \dots, x_n)$, where x_i maps to 0 under the projection to the i-th factor. We can see that all these elements are indeed in B. □

2. Let $M = \mathbb{Z}^2$ be the free $\mathbb{Z}$-module of rank 2. Let N be the submodule $24\mathbb{Z} \oplus 30\mathbb{Z}$. Find a basis y_1, y_2 for M and integers a_1, a_2 such that $a_1 \mid a_2$ and a_1y_1, a_2y_2 is a basis for N. Deduce that $\mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/30\mathbb{Z}$ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/120\mathbb{Z}$.

   Proof. Following the proof of the fundamental theorem of Principal Ideal domains we will get that $y_1 = (4, 5)$ and $y_2 = (1, 1)$ and $a_1 = 6$ and $a_2 = 120$. Using this and the first question shows the desired isomorphism. □

3. Let $R = \mathbb{Z}[x]$, let M be the free module of rank 1. Consider the submodule N generated by 2 and x. Show that the rank of N is 1. Show that N is not a free module of rank 1 (This is equivalent to N being a principal ideal).

   Proof. Pick any two elements f, g of N, then $gf - fg = 0$ is a linear combination where neither coeffecient is 0, thus the rank is less than 2. The rank of an any ideal in an integral domain is at least 1 so the rank of N is exactly 1.

   Considering N as an ideal we see that being a free module of rank 1 is equivalent to N being a principal ideal. We have already seen that this is not true. □

4. Show that an integral domain R is a PID iff every submodule of a free module is a free module. (Note we proved one direction in class. For the other direction consider cyclic modules.)

   Proof. One direction was proved in class.

   For the other direction the argument above will show that any submodule of a free module of rank 1 has rank bounded by 1. Thus were it to be a free module it would be a free module of rank 1. I.e. every ideal is principal. Thus if R is not a principal ideal domain, then a non-principal ideal gives a module of rank 1 which cannot be free of rank 1 since it is not generated by any 1 element. □

5. Let $V = \mathbb{R}^2$ with basis e_1, e_2. Show that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes_{\mathbb{R}} V$ cannot be written as any element $v \otimes w$.

   Proof. This is similar to the example shown in class and the proof is exactly analogous. □

6. Let $V = \mathbb{R}^n$ nd let v, v' be non zero elements. Prove that $v \otimes v' = v' \otimes v$ in $V \otimes_{\mathbb{R}} V$ if and only if $v = av'$ for some $a \in \mathbb{R}$.

   Proof. By definition it is true that if $v = av'$, then $v \otimes v' = v' \otimes v$.

   For the other direction, if there is no a such that $v = av'$, then we can form a basis containing both v and v'. If $v \otimes v' = v' \otimes v$, then $v \otimes v' + v' \otimes v$ can be written as a basic tensor. The same proof from class used in the previous questions shows that this cannot happen. □

7. Show that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ is isomorphic to $\mathbb{C}$.

   Proof. Define a map $B\colon \mathbb{Z}[i] \times \mathbb{R} \to \mathbb{C}$ by $B(x, y) = xy$. This is a bilinear map and by the universal property we get a map $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ to $\mathbb{C}$. To check that it is an isomorphism we define a map ψ in the other direction by $\psi(a + bi) = (1 \otimes a) + (i \otimes b)$. These two maps are mutually inverse and so define isomorphisms. □

8. Show that $\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}[x]$

   Proof. As above define a map $B\colon \mathbb{Z}[x] \times \mathbb{Q} \to \mathbb{Q}[x]$ by $B(f(x), q) = qf(x)$. This is a bilinear map so we get a well defined homomorphism $\phi\colon \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}[x]$. To prove that it is an isomorphism we can create and inverse namely the map which sends $\sum_{i=0}^{n} a_i x^i$ to $\sum_{i=0}^{n} x^i \otimes a_i$. These are mutually inverse and so define isomorphims. □

9. Show that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is an $\mathbb{R}$-module via $r(c \otimes c') = (rc) \otimes c'$. Show that it is a free R-module of rank 2. Show that the $\mathbb{R}$-module $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has rank > 2.

Proof. We know that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is $\mathbb{C}$ module and the multiplication defined above is just the restriction of this to $\mathbb{R}$. Thus this is a well defined $\mathbb{R}$-module.

We also know this module is isomorphic to $\mathbb{C}$ as a $\mathbb{C}$-module and since the multiplication defined is just a restriction and $\mathbb{C}$ is a free $\mathbb{R}$-module of rank 2 generated by 1 and i. Thus, $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is a free module of rank 2.

From class we can show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a free module of rank 4. This is done using the fact that $\mathbb{C}$ is a free module of rank 2. □

10. Let R be an integral domain contained in a field Q. Let M be a torsion R module. Show that $M \otimes_R Q = 0$.

Proof. It suffices to show that basic tensors are all 0. Given $m \in M$ let $r_m \in R \setminus \{0\}$ be such that $r_m m = 0$. Let $m \otimes q$ be a basic tensor since Q is a field containing R we can write q as $r_m(\frac{1}{r_m})$. We can then move the r_m across the tensor product to get the following chains of equalities completing the proof.

$$m \otimes q = m \otimes r_m(\frac{1}{r_m}) = r_m m \otimes (\frac{1}{r_m}) = 0 \otimes (\frac{1}{r_m}) = 0.$$

□

11. Let R be an integral domain, Q the field of fractions, M an R-module. Show that every element of $M \otimes_R Q$ can be written as $(1/d) \otimes c$ for some $c \in M, d \in R$.

Proof. This is a clever use of common denominators let $x \in M \otimes_R Q$ be given by $x = \sum_{i=1}^n m_i \otimes q_i$. Since $q_i = \frac{r_i}{s_i}$ we can write all of these over a common denominator namely

$$q_i = \frac{r_i}{s_i} = \frac{r_i \prod_{j \neq i} s_j}{\prod_{j=1}^n s_j}.$$

Further every numerator is an element of R so we can do the following:

$$m_i \otimes q_i = m_i \otimes \frac{r_i \prod_{j \neq i} s_j}{\prod_{j=1}^n s_j} = \left(r_i \prod_{j \neq i} s_j m_i \right) \otimes \left(\frac{1}{\prod_{j=1}^n s_j} \right).$$

We can now add these together to get:

$$x = \sum_{i=1}^n \left(\left(r_i \prod_{j \neq i} s_j m_i \right) \otimes \left(\frac{1}{\prod_{j=1}^n s_j} \right) \right) = \left(\sum_{i=1}^n r_i \prod_{j \neq i} s_j m_i \right) \otimes \left(\frac{1}{\prod_{j=1}^n s_j} \right).$$

□

The following questions are for extra credit!!!

12. Show that $\bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ is a torsion $\mathbb{Z}$-module but $\prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ is not a torsion $\mathbb{Z}$-module by showing that the element $(1, 1, \dots)$ has infinite order.

    Proof. Given an element $x = (x_1, x_2, \dots)$ let $m \in \mathbb{N}$ be such that $x_i = 0$ for all $i > m$. The order of x is now bounded by 2^m hence this is a torsion module.

    That $m \cdot (1, 1, \dots) = (m, m, \dots)$ if this were 0, then m would have to be divisible by 2^i for all i and hence $m = 0$. Thus this element has infinite order and $\prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ is not a torsion $\mathbb{Z}$-module. □

13. Let R be an integral domain and Q the field of fractions. Show that if N is a submodule of an R-module M, then $N \otimes_R Q$ is a submodule of $M \otimes_R Q$.

    Proof. Define the map $B\colon N \times Q \to M \otimes_R Q$ by $B(n, q) = n \otimes q$. This is a bilinear so defines a homomorphism from $N \otimes_R Q$ to $M \otimes_R Q$. This homomorphism is injective, since since it is injective on basic tensors and every element can be written as a basic tensor by a previous question. □

14. Deduce from the above that tensor product does not commute with direct products. Namely that $(\prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \neq \prod_{i=1}^{\infty}(\mathbb{Z}/2^i\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q})$

    Proof. We know that $\prod_{i=1}^{\infty}(\mathbb{Z}/2^i\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$ since each summand is 0 by a previous question.

    Let N be the submodule of $\prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ generated by $(1, 1, \dots)$ this is a copy of $\mathbb{Z}$ and so $N \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ is a submodule of $(\prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and so this module is non-zero. □