Algebra 145 Lecture Notes

Robert Kropholler

December 10, 2018

1. Syllabus, Sets and functions, Equivalence relations
2. Modular arithmetic
3. Binary Operations and Group Axioms
4. Cayley Tables and Cyclic Groups
5. Dihedral Groups
6. Cayley tables for Dihedral groups, Symmetric groups and permutations
7. Cycle notation, cycle type and transpositions
8. Even and odd permutati ons, the alternating group
9. Subgroups generated by a set
10. Orders of elements, Lagrange's theorem and consequences
11. Homomorphisms, isomorphisms
12. Kernels and images, Normal subgroups
13. Quotient groups
14. Well definedness for maps and review
15. Midterm
16. Isomorphism theorem
17. Finitely generated abelian groups
18. Group Actions and Cayley's theorem
19. Orbit-Stabiliser theorem
20. Burnside's Lemma
21. Simplicity of the alternating group (not examinable)

22. Puzzles (Not examinable)

23. Ring theory

24. More ring theory

25. Polynomial Rings

26. Ideals and Homomorphisms

27. Isomorphism theorems and chinese remainder theorem

28. Review/overflow

29. Review/overflow
    FINAL

Contents

1 The Preliminaries

1.1 Notations

Mathematics is a language and like any other language comes with a set of rules and shorthand notations. While I will try and keep use of these abbreviations to a minimum we should all understand the following.

1.2 Basic logic

If P and Q are two statements, then $P \Rightarrow Q$ means that if P is true, then Q is true. In this case, we say that P implies Q.

For instance, if x is odd, then $x \neq 2$ or if Ron is a cat, then Ron is not a dog.

If $P \Rightarrow Q$ and $Q \Rightarrow P$, then we write $P \Leftrightarrow Q$, we say P is true if and only if Q is true. For instance, x is an even prime if and only if $x = 2$.

The symbol $\forall$ should be read as "for all". The symbol $\exists$ should be read as "there exists", $\exists!$ should be read as "there exists a unique".

1.3 Set notation

Let S and T be two sets.

If s is an element of S, then we write $s \in S$ and similarly $s \notin S$ is used to denote that s is not a member of S. For instance $2 \in \mathbb{Z}$ and $\frac{1}{2} \notin \mathbb{Z}$.

If S has finitely many elements, then we say that S is a finite set. We write $|S|$ to denote the cardinality of S i.e. the number of elements in S.

The standard way of writing a set is to specify what the elements look like and then what properties they have. We do this with braces $\{,\}$ and $|$ the latter should be read as such that. For instance, $\{x \in \mathbb{Z} \mid 2 \text{ divides } x\}$, this is the set of even integers.

If the set is finite, then we can just write the elements out. For instance, $\{0, 1, 2, 3, 4, 5\}$ is the set containing the first five natural numbers. This is equivalent to writing $\{n \in \mathbb{N} \mid 0 \leqslant n \leqslant 5\}$.

If every object of S is also an object of T, then we say that S is a subset of T and write $S \subset T$. If $S \subset T$ and $T \subset S$, then $S = T$.

For the complement of a set S in a set T we write $T \smallsetminus S$. This is the set $\{x \in T \mid x \notin S\}$.

The intersection is denoted $S \cap T$ and is the set $\{x \mid x \in S \text{ and } x \in T\}$.

The union of two sets, denoted $S \cup T$ is the set $\{x \mid x \in S \text{ or } x \in T\}$.

The cartesian product of S and T, denoted $S \times T$ is the set of ordered pairs of elements of S and T. Formally, $S \times T = \{(a, b) \mid a \in S, b \in T\}$.

A function f is a mapping from a set S to another set T. We express this using the following notation.

$$\begin{aligned} f \colon S &\to T \\ x &\mapsto f(x) \end{aligned}$$

Some examples of this notation:

$$\begin{aligned} f \colon \mathbb{N} &\to \mathbb{N} \\ x &\mapsto x^2 \end{aligned}$$

$$\begin{aligned} f \colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \cos(\sin(x)) \end{aligned}$$

Definition 1.3.1. Let S and T be sets and $f \colon S \to T$ be a map.

- We say S is the *domain* of f and T is the *codomain* of f.
- We say that f is the identity is $S = T$ and $f(x) = x$ for all $x \in S$. We denote the identity by id_S.
- We say that f is *injective* if $f(x) = f(y) \Rightarrow x = y$.
- We say that f is *surjective* if given a $t \in T$ there is and $s \in S$ such that $f(s) = t$.
- A map is *bijective* if it is both injective and surjective.
- If R is a set and $g \colon R \to S$ is a map, then we can compose f and g, denoted $f \circ g$. This is a map from R to T.

1.4 Relations

Within language we naturally talk about people or objects being related. We can do this abstractly in the world of sets. A relation on a set is a way of grouping objects of a set that are similar to one another. The following is the formal definition, however thinking naively works perfectly well.

Definition 1.4.1. A *relation* R on a set S is a subset of $S \times S$. One usually writes aRb if $(a, b) \in R$ and says a is related to b.

For instance, a relation on the set $\mathbb{Z}$ could be aRb if a is even. A relation on $\mathbb{R}$ is given by aRb if $a = b^3$.

There are certain extra properties that one might wish to put on a relation.

Definition 1.4.2. Let S be a set and R be a relation on S.

- We say R is *reflexive* if $(a, a) \in R$ for every $a \in S$.
- We say R is *symmetric* if $(a, b) \in R \Rightarrow (b, a) \in R$.
- We say R is *transitive* if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

We say that R is an *equivalence relation* if it is reflexive, symmetric and transitive. Equivalence relations will usually be denoted by $\sim$

Example. $S = \mathbb{R}$ and xRy if and only if $x^4 = y^4$. This is an equivalence relation.

Example. $S = \mathbb{R}$ and xRy if and only if $y \leqslant x^2$. This is transitive.

Example. $S = \mathbb{Z}$ and xRy if and only if $y \leqslant x^2$. This is reflexive and transitive.

Example. $S = \mathcal{P}(\mathbb{N})$ and XRY if and only if the smallest element of X is equal to the smallest element of Y. This is an equivalence relation.

Example. $S = \mathbb{Z} \times \mathbb{Z}$ and $(a, b)R(c, d)$ if and only if $a = c$ and $b^2 = d^2$. This is an equivalence relation.

Example. $S = \{\text{words in the English language}\}$ and wRv if the are synonymous. This is reflexive and symmetric.

Example. $S = \mathbb{Z}$ and xRy if and only if 2 divides x^2 and 2 divides y^2. This is an symmetric and transitive.

Most relations that you can think of will be equivalence relations although coming up with examples that don't satisfy at least one of these properties is not hard.

Definition 1.4.3. Let $\sim$ be an equivalence relation on a set S. Then we define the *equivalence class* of x as

$$[x] = \{y \in S \mid x \sim y\}.$$

Since equivalence relations are reflexive, we see that $x \in [x]$, this also tells us that the union of equivalence classes is the set S.

Using the symmetric and transitive property, it can be shown that either $[x] = [y]$ or $[x] \cap [y] = \varnothing$.

Definition 1.4.4. Let S be a set and $\{X_i\}$ be a collection of subsets of S. We say that $\{X_i\}$ is a partition of S if the following hold:

- Each $X_i \neq \varnothing$.
- The union of the X_i is S.
- If $i \neq j$, then $X_i \cap X_j = \varnothing$.

We have seen that the equivalence classes of an equivalence relation form a partition. These concepts will appear many times throughout the course so I recommend becoming familiar with them as quickly as possible.

1.5 Modular Arithmetic

Fix an $n \in \mathbb{N}$ given any integer m there is a remainder r when divided by n. This remainder satisfies the following the inequality $0 \leqslant r < n$. We call r the *remainder modulo* n. This gives a natural equivalence relation on $\mathbb{Z}$:

$$a \sim b \Leftrightarrow a - b \text{ is divisible by } n \Leftrightarrow a \text{ and } b \text{ have the same remainder modulo } n.$$

We denote the equivalence classes of this equivalence relation by $\mathbb{Z}/n\mathbb{Z}$. There are n equivalence classes $[0], [1], \ldots, [n-1]$. We call these equivalence classes *residue classes*.

There is a surjective map $[\cdot]\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ given by $m \mapsto [m]$. This map is clearly not injective, e.g. $m, m+n$ have the same image.

One can describe the residue classes in the following way $[m] = \{l \in \mathbb{Z} \mid l = m + kn \text{ for some } k \in \mathbb{Z}\}$. Using this one can prove the following.

Proposition 1.5.1. *Let $n \in \mathbb{N}$ and $a, a', b, b' \in \mathbb{Z}$. If $[a] = [a']$ and $[b] = [b']$, then $[a+b] = [a'+b']$ and $[ab] = [a'b']$.*

This allows us to define addition and multiplication on $\mathbb{Z}/n\mathbb{Z}$ by $[a] + [b] = [a+b]$ and $[a] \times [b] = [ab]$.

This definition seems to depend on choice of $a \in [a]$ and $b \in [b]$ although the above proposition tells us that if we pick different representatives, then we obtain the same result.

This addition and multiplication work similarly to that of $\mathbb{Z}$. For instance there is a zero, i.e. $[0]$ with the property that $[a] + [0] = [a] = [0] + [a]$.

There is also an analogue of unity given by $[1]$. This has the property that $[1] \times [a] = [a] = [a] \times [1]$.

Also notice, similarly to $\mathbb{Z}$, that $[0] \times [a] = [0] = [a] \times [0]$.

We call a residue class $[a]$ *non-zero* if $[a] \neq [0]$.

Some properties that differ from $\mathbb{Z}$ are that we can add $[1]$ to itself repeatedly and eventually get back to $[0]$. Also if $n = rs$ where $1 < r, s < n$, then $[r] \times [s] = [0]$. One property of $\mathbb{Z}$ is that if $xy = 0$, then $x = 0$ or $y = 0$.

Proposition 1.5.2. *For every $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ the congruence*

$$mx \equiv 1 \mod n$$

has a solution if and only if m and n are coprime.

Proof. If m and n are coprime, then by the Euclidean algorithm we can find r, s such that $rm + sn = 1$ thus r is the solution to the stated equation.

If there is a solution to the equation, then there is an integer r such that $rm = 1 + sn$. Thus $rm - sn = 1$, the greatest common divisor of m and n divide the left hand side and thus must be equal to 1. Hence m and n are coprime. □

The congruence above can be written as an equation in $\mathbb{Z}/n\mathbb{Z}$ by $[a] \times [x] = [1]$. We say that $[a]$ has a multiplicative inverse if there exists a $b \in \mathbb{Z}$ such

that $[a] \times [b] = [1]$. The above proposition shows that $[a]$ has a multiplicative inverse if and only if a and n are coprime.

In $\mathbb{Q}$ multiplication has the property that for all non-zero rational numbers q there is a rational number r such that $qr = 1$. We can examine when this property holds for $\mathbb{Z}/n\mathbb{Z}$. This is the same as asking that n is coprime to $1, 2, \ldots, m-1$. This occurs only when n is prime.

Later we will phrase such objects as fields.

Not all objects we meet will come from numbers many will be more abstract and exotic but these form a very nice class of examples that will appear often throughout the course.

2 Group Theory

We begin by studying groups in some ways these are the simplest of algebraic objects. They have a single binary operation which satisfies some additional properties.

2.1 Binary Operations

Definition 2.1.1. Let G be a set. A *binary operation* $*$ on G is a map $*\colon G \times G \to G$. We write $a * b$ for the image of (a, b).

A binary operation is a way of defining a multiplication on a set. It takes in 2 elements of the set and outputs a third. You are already familiar with several binary operations, although they may not be familiar in such terminology.

Examples.

1. Addition, $+$, on the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.
2. Subtraction, $-$, on the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.
3. Multiplication, $\times$, on the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.
4. Matrix multiplication.
5. Addition of vectors in a vector space.
6. Cross product of vectors in $\mathbb{R}^3$.
7. Multiplication of polynomials.
8. Composition of functions $\mathbb{R} \to \mathbb{R}$.

Non Example. Division, $\div$, in $\mathbb{R}$. This is not a binary operation as $1 \div 0$ is not defined.

Many of the binary operations you have met satisfy the property of being associative.

Definition 2.1.2. A binary operation is *associative* if

$$a * (b * c) = (a * b) * c.$$

Hence, the expression $a * b * c$ has a well defined meaning.

Exercise. Which of the above binary operations are associative?

With associativity we can drop all brackets from expressions, i.e. $a_1 * a_2 * \cdots * a_n$ has a well defined meaning.

Another nice property that one might care about is commutativity.

Definition 2.1.3. A binary operation is *commutative* if

$$a * b = b * a.$$

Exercise. Which of the above binary operations are commutative?

Definition 2.1.4. An element $e \in G$ is an *identity* for the binary operation $*$ if, for any $a \in G$,

$$a * e = a = e * a.$$

Exercise. Which of the above binary operations have an identity? What is the identity in these cases?

Proposition 2.1.5. *If a binary operation has an identity e, then it is unique.*

Proof. Let e_1 and e_2 be identities, then the following equalities hold:

$$\begin{aligned} e_1 * e_2 &= e_2 \qquad \text{as } e_1 \text{ is an identity.} \\ e_1 * e_2 &= e_1 \qquad \text{as } e_2 \text{ is an identity.} \end{aligned}$$

So $e_1 = e_2$. □

Definition 2.1.6. If a binary operation $*$ has an identity $e \in G$ and $a \in G$, then $b \in G$ is said to be an *inverse* of a if

$$a * b = e = b * a.$$

Proposition 2.1.7. *Let $*$ be an associative binary operation on a set G with an identity e and let $a \in G$. Then an inverse of a, if it exists, is unique.*

Proof. Let $b, c \in G$ be elements such that $a * b = e = c * a$. Consider the element of G given by $c * a * b$ on since $c * a = e$, $e * b = b$ and $*$ is associative we arrive at $c * a * b = b$. Also, since $a * b = e$, $c * e = c$ and $*$ is associative we arrive at the equality $c * a * b = c$. Thus $b = c * a * b = c$. □

Notation 2.1.8. We denote the inverse of a, if it exists, a^{-1}.

Let G be a set and let $*$ be a binary operation on G. If H is a subset we can restrict $*$ to H to obtain a map $*\colon H \times H \to G$. This will not usually be a binary operation on H. If the binary operation restricts to a binary operation on H, then we say H is *closed under* $*$.

Examples. $n\mathbb{Z}$ and $\mathbb{N}$

2.2 Group Axioms

Definition 2.2.1. A *group* $(G, *)$ consists of a set G and a binary operation $*$ on G satisfying the following axioms.

- The binary operation $*$ is associative.
- There is an identity element $e \in G$.
- For each $a \in G$, there is an inverse, a^{-1}, for a.

Remark 1. When the binary operation $*$ is clear from context we will simply write "G is a group" as a shorthand for "$(G, *)$ is a group"

When verifying that $(G, *)$ is a group we must check the three axioms above as well as the fact that $*$ is a binary operation on the set G. This is sometimes to referred to as closure of the operation.

As in basic algebra we regularly suppress $*$ in notation. Thus, $a * b$ is simply written as ab.

Let n be an integer. We will use the following shorthand:

$$x^n = \begin{cases} \underbrace{xx\dots x}_{n \text{ times}} & n > 0, \\ e & n = 0, \\ \underbrace{x^{-1}x^{-1}\dots x^{-1}}_{-n \text{ times}} & n < 0. \end{cases}$$

One can quickly check that the following come directly from the axioms of being a group.

Proposition 2.2.2. *Let x, y be elements of a group G and let n, m be integers.*

1. $(xy)^{-1} = y^{-1}x^{-1}$.
2. $x^n x^m = x^{n+m}$
3. $(x^n)^m = x^{nm}$
4. *If $xz = xy$, then $z = y$.*
5. *If $zx = yx$, then $z = y$.*

Proof. Exercise □

Definition 2.2.3. We say that a group $(G, *)$ is *abelian* if $*$ is a commutative binary operation. I.e. $xy = yx$ for all $x, y \in G$.

Example. The set $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ with addition as a binary operation form abelian groups. In each case $e = 0$ and $x^{-1} = -x$.

Example. The sets $\mathbb{Q} \setminus \{0\}, \mathbb{R} \setminus \{0\}, \mathbb{C} \setminus \{0\}$ with multiplication form abelian groups. In each case $e = 1$ and $x^{-1} = \frac{1}{x}$.

Example. The set $(0, \infty)$ of positive real numbers under multiplication forms a group. Once again $e = 1$ and $x^{-1} = \frac{1}{x}$.

Example. Any vector space with the operation of vector addition forms a group.

Example. The set of real invertible $n \times n$ matrices under matrix multiplication form a group. This group is the *general linear group* $GL_n(\mathbb{R})$.

There are many other interesting groups of matrices, here are a few.

Example. The set of real $n \times n$ matrices with determinant 1 under matrix multiplication. This is the *special linear group* $SL_n(\mathbb{R})$.

Example. The set of $n \times n$ orthogonal matrices under matrix multiplication forms the group $O(n)$. (Recall a matrix A is orthogonal if $A^{-1} = A^T$.)

Example. The set of $n \times n$ orthogonal matrices with determinant 1 under matrix multiplication forms the group $SO(n)$.

Example. The set $\mathbb{Z}/n\mathbb{Z}$ forms a group under addition.

Example. The set of elements coprime to n in $\mathbb{Z}/n\mathbb{Z}$ forms a group under multiplication.

Example. We can look at symmetries of objects. For instance the triangle or a rectangle.

Example. Given any set S we can look at the set of bijections $S \to S$. This forms a group denoted $Sym(S)$.

Finally we can build new groups from old ones by taking direct products.

Theorem 2.2.4. *Let $(G, *_G)$ and $(H, *_H)$ be groups. The the operation $*$ on $G \times H$ given by*

$$(g, h) * (g', h') = (g *_G g', h *_H h')$$

is a group operation. The group $(G \times H, *)$ is called the direct product of G and H.

Proof. The binary operation $*$ is associative since both $*_G$ and $*_H$ are associative.

The identity for $*$ is (e_G, e_H) where e_G is the identity in G and e_H is the identity in H.

The inverse of the element (g, h) is the element (g^{-1}, h^{-1}).

Thus the operation $*$ satisfies the three axioms of a group and $(G \times H, *)$ forms a group. □

2.2.1 Cayley tables

Let G be a group. Each $g \in G$ gives a map from $\alpha_g \colon G \to G$ given $\alpha_g(h) = gh$. We can record this information in a table with G rows and G columns. In this (g, h) place we put gh. Since $gh = gh'$ implies that $h = h'$ we see that each row has every element of G exactly once. Similarly each column has each element of G exactly once.

An $n \times n$ grid filled with n symbols such that each symbol occurs once in each column and each row is called a *Latin square*. You have probably come across Latin squares in the form of Su Doku or other number puzzles.

Not all Latin squares correspond to groups but when they do they are referred to as *Cayley tables*.

Lets look at some examples.

We have already seen the group $\mathbb{Z}/5\mathbb{Z} = \{[0], [1], [2], [3], [4]\}$ and we can write the Cayley table as follows:

$\cdot$	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

Table 1: A Cayley table for $\mathbb{Z}/5\mathbb{Z}$.

Non Example. Given 5 elements x, y, z, t, e we can get the following Latin square which does not correspond to any group.

$\cdot$	e	x	y	z	t
e	e	x	y	z	t
x	x	e	t	y	z
y	y	t	z	e	x
z	z	y	x	t	e
t	t	z	e	x	y

How do we know this doesn't correspond to a group. Since $ty = e$, we have that $t = y^{-1}$ in a group this would imply that $yt = e$ however the table tells us that $yt = x$.

We can deduce many things about a group from its Cayley table. Firstly we can see if the group is abelian. If the elements of the group are labelled x_i, then the (i, j)-th position of the Cayley table is $x_i x_j$. Thus we can see if the group is abelian by checking that the (i, j)-th entry and the (j, i)-th entry are the same. Alternatively we can see if the Cayley table is symmetric about the leading diagonal.

Definition 2.2.5. We say that two Cayley tables are *equivalent* if there is a way to reorder the elements of the group such that the rows and column of one are the same as the rows and columns of the other.

Later we will formalise this to the notion of isomorphism but this will do for now.

Proposition 2.2.6. *If two Cayley tables are equivalent, then the groups they represent are the same with the elements reordered.*

This allows us to assume that the first column and row correspond to the identity element of the group.

Let us look at Cayley tables for a group of with 3 elements, let us label these elements $\{e, x, y\}$.

We can start filling it out to get the following.

$\cdot$	e	x	y
e	e	x	y
x	x		
y	y	$\square$	

Let us now consider the entry labelled by $\square$ in the above table. This cannot be x or y since they already appear in that column or row respectively. So we see that this entry must be e and we get the following,

$\cdot$	e	x	y
e	e	x	y
x	x	$\circ$	
y	y	e	

We now know that the $\circ$ in the above table must be y and this allows us to fill in the rest of the table to obtain:

$\cdot$	e	x	y
e	e	x	y
x	x	y	e
y	y	e	x

This may not be a table which represents a group however if we compare it to the table for $\mathbb{Z}/3\mathbb{Z}$ i.e.

$\cdot$	$[0]$	$[1]$	$[2]$
$[0]$	$[0]$	$[1]$	$[2]$
$[1]$	$[1]$	$[2]$	$[1]$
$[2]$	$[2]$	$[1]$	$[0]$

We see that the table above does indeed represent a group. In particular there is only one group with 3 elements.

Let us look at groups with 4 elements $\{e, x, y, z\}$. Once again the first row and column are easy and we start with,

$\cdot$	e	x	y	z
e	e	x	y	z
x	x			
y	y			
z	z			

We now have a choice for the xy entry, it can be either e or z. Let us begin with the case that it is e. We also know that if $xy = e$, then $yx = e$. So we get,

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	?	e	?
y	y	e	?	?
z	z	?	?	?

Since there is an e in each column and row, we see that $z^2 = e$. Also focusing on the third row and column we see that $yz = zy = x$. We now get,

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	?	e	?
y	y	e	?	x
z	z	?	x	e

We can now deduce the rest to obtain

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	z	e	y
y	y	e	z	x
z	z	y	x	e

If we now replace e with [0], x with [1], t with [3] and z with [2] we obtain the table for $\mathbb{Z}/4\mathbb{Z}$.

$\cdot$	[0]	[1]	[3]	[2]
[0]	[0]	[1]	[3]	[2]
[1]	[1]	[2]	[0]	[3]
[3]	[3]	[0]	[2]	[1]
[2]	[2]	[3]	[1]	[0]

We see that this is certainly the table for a group and is in fact the table for $\mathbb{Z}/4\mathbb{Z}$.

Now we return to the case that $xy = z$. Since $xy = e$ if and only if $yx = e$, we also know that $yx = z$. We obtain the following

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	?	z	?
y	y	z	?	?
z	z	?	?	?

We now once again have two options. Either $x^2 = e$ or $x^2 = y$. Let us begin with the case that $x^2 = y$ filling in the obvious blanks we obtain the following.

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	y	z	e
y	y	z	?	?
z	z	e	?	?

We also see that $zy = x$ and from this we can finish the table.

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	y	z	e
y	y	z	e	x
z	z	e	x	y

Once again we can ask if this is a group table and indeed by relabelling e as [0], x as [1], y as [2] and z as [3] we once again obtain the table for $\mathbb{Z}/4\mathbb{Z}$.

$\cdot$	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

We now have one more case $xy = z$ and $x^2 = e$, we also deduce that $xz = y = zx$ we obtain the following

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	?	?
z	z	y	?	?

We now once again split into two options. If we let $y^2 = z$, then we can replace to once again obtain the table for $\mathbb{Z}/4\mathbb{Z}$ again. If we let $y^2 = e$ we obtain the following:

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

This is a fundamentally different group but is the same as the group of symmetries of a rectangle discussed earlier. This group is known as the Klein 4 group. What we have seen is that all groups with 4 elements are abelian.

2.2.2 Cyclic groups

Definition 2.2.7. A group G is *cyclic* if there is an element $g \in G$ such that $G = \{g^k \mid k \in \mathbb{Z}\}$.

Such a g is called a *generator* for G.

Cyclic groups form and important class of groups.

Example. The group $\mathbb{Z}$ is a cyclic group with generator 1 or -1.

Definition 2.2.8. The cardinality $|G|$ of a group G is called the *order* of G. We say that a group is *finite* if $|G|$ is finite.

Definition 2.2.9. The *cyclic group of order* n, C_n, is the group with elements

$$e, g, g^2, \ldots, g^{n-1}$$

which satisfy $g^n = e$. Thus the multiplication is defined by

$$g^i * g^j = \begin{cases} g^{i+j} & \text{if } 0 \leqslant i + j < n, \\ g^{i+j-n} & \text{if } n \leqslant i + j \leqslant 2n - 2. \end{cases}$$

Definition 2.2.10. Let g be an element of a group G. The *order of* g is the minimal $n > 0$ such that $g^n = e$. If no such n exists we say that g is of infinite order.

We denote the order of g by $o(g)$.

Lemma 2.2.11. *Let g be an element of a group with finite order. Then $g^k = g^{k-o(g)}$. Also, $g^k = g^{k+o(g)}$.*

Proof. Exercise. Key ingredient: $g^{o(g)} = e$. □

Theorem 2.2.12. *Let g be an element of a group. Then $g^k = e$ if and only if $o(g)$ divides k.*

Proof. For one direction, if $o(g)$ divides k, then $k = p(o(g))$. Thus, $g^k = (g^{o(g)})^p = e^p = e$.

For the other direction, if $g^k = e$, then using Lemma 2.2.11 we can repeatedly add or subtract $o(g)$ from k to obtain l. Thus l has the following properties:

- $g^k = g^l$.
- $l = k + q(o(g))$ for some $q \in \mathbb{Z}$.
- $0 \leqslant l < o(g)$.

Since $o(g)$ was minimal among n such that $n > 0$ and $g^n = e$, we see that $l = 0$. Thus, $0 = k + q(o(g))$ and k is divisible by $o(g)$. □

Proposition 2.2.13. *If g is an element of a group G, then the order of g is the size of the set $\{g^k \mid k \in \mathbb{Z}\}$.*

Proof. Suppose that g has infinite order, we will show the set is infinite. It contains the subset $\{g^k \mid k \in \mathbb{N}\}$. Suppose that two elements of this set are equal i.e. there are positive integers k, l such that $g^k = g^l$. We can assume that $k < l$. So $g^{l-k} = g^l g^{-k} = e$. Thus contradicting g having infinite order. Thus the set $\{g^k \mid k \in \mathbb{Z}\}$ is an infinite set.

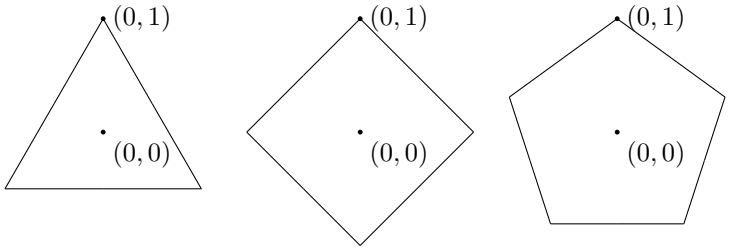
Suppose now that g has finite order. By Lemma 2.2.11 we can replace any power of g with g^l such that $0 \leqslant l < o(g)$. Thus $\{g^k \mid k \in \mathbb{Z}\} = \{g^k \mid 0 \leqslant k < o(g)\}$ so we have that $|\{g^k \mid k \in \mathbb{Z}\}| \leqslant o(g)$. Suppose there were two numbers $0 \leqslant k, l < o(g)$ such that $g^k = g^l$. Then $g^{k-l} = e$ we may assume that $k - l \geqslant 0$. Thus, $0 \leqslant k - l < o(g)$ so by minimality of $o(g)$ we have that $k - l = 0$. Thus, $|\{g^k \mid 0 \leqslant k < o(g)\}| = o(g)$. □

Corollary 2.2.14. *If G is finite, then $o(g)$ is finite for all elements $g \in G$.*

Proof. The set $\{g^k \mid k \in \mathbb{Z}\}$ is a subset of G which is a finite group. □

2.2.3 Dihedral groups

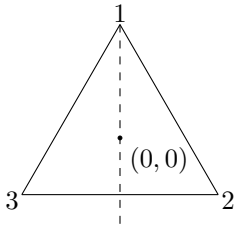
Let $n \geqslant 3$ be an integer. Let P be the regular n-gon in the plane with center at $(0, 0)$ and a vertex at $(0, 1)$. The first few are depicted below



Definition 2.2.15. The *n-th dihedral group* D_n is the group of linear maps f from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $f(P) = P$. This is the set of symmetries of a regular n-gon. This is a group since:

- The identity map is in D_n.
- The composition of two such maps is in D_n.
- The inverse of such a map is in D_n
- Composition of functions is associative.

Let us carefully study some small cases. First consider the case $n = 3$.



Let r be the rotation clockwise by $\frac{2\pi}{3}$ and let s be the reflection in the dashed line through $(0,1)$ and $(0,0)$. We will also label the vertices $1,2,3$. We have the following six symmetries

$$e, r, r^2, s, rs, r^2s.$$

These are all different since they each do different things to the vertices. In fact, they give the following permutations:

$$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}.$$

In fact, since these are the only possible permutations of the vertices, these must be all the elements.

In the case $n = 4$, we similarly let r be the rotation clockwise by $\frac{\pi}{2}$ and s as before. We now have the following 8 elements along with the permutation they induce on the vertices

$$e = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}, \quad r = \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}, \quad r^2 = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}, \quad r^3 = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix},$$
$$s = \begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}, \quad rs = \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}, \quad r^2s = \begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}, \quad r^3s = \begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}.$$

Once again we can see that these are 8 different elements. However, it is harder to see that these are all the elements as there are $4! = 24$ permutations of the vertices. We do however have the following.

Proposition 2.2.16. *There are $2n$ elements of D_n.*

Furthermore, let r be the rotation clockwise through an angle of $\frac{2\pi}{n}$. Let s be the reflection through the y-axis. Then D_n consists of the elements $e, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s$.

Proof. Labelling the vertices of the regular n-gon with the numbers $1, \ldots, n$ we see that any element of D_n send the vertex labelled 1 to any other vertex. Once one has decided where 1 goes then there are 2 choices for where the map sends 2. Once the images of 1 and 2 are chosen, then the image of 3 is decided since this must be adjacent to 2 but cannot be the image of 1. Similarly the image of all the other vertices are decided. This gives at most $2n$ elements of D_n.

To check that there are in fact $2n$ elements, consider the list of elements given by $e, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s$. One can check that these elements send

the vertices labelled 1 and 2 to any choice of 2 adjacent vertices in the polygon P. Thus these are all $2n$ possible choices of map. □

Proposition 2.2.17. *The following identities hold in D_n.*

$$r^n = e$$
$$s^2 = e$$
$$sr = r^{n-1}s = r^{-1}s$$

Proof. One can simply check that the maps above are as stated. □

2.3 Symmetric groups

Definition 2.3.1. Let S be a set. A bijection $f\colon S \to S$ is called a *permutation* of S and the set of all permutations of S is called $Sym(S)$.

If n is a positive integer, then we write S_n for $\mathrm{Sym}(\{1, 2, \ldots, n\})$.

Theorem 2.3.2. *Let S be a set.*

1. *Then* $\mathrm{Sym}(S)$ *is a group, where the binary operation is composition of functions.*
2. *The order of S_n is $n!$.*
3. *If $|S| > 2$, then* $\mathrm{Sym}(S)$ *is non-abelian.*

Proof. For the first point it is clear that the composition of two bijections is a bijection, thus this is binary operation on $\mathrm{Sym}(S)$. Composition of functions is an associative operation. The map id_S is the identity for this operation. We proved on Problem sheet 1 Q3, that any bijective map has an inverse. Thus $\mathrm{Sym}(S)$ is a group.

For a bijection f on $\{1, 2, \ldots, n\}$, we have n possibilities for $f(1)$. For $f(2)$ we have $n-1$ choices since we can choose any element which isn't $f(1)$. Repeating this we make $n!$ choices. Thus there are $n!$ elements of S_n.

To show that it is not Abelian consider the permutation σ which exchanges 1 and 2 and the permutation τ which exchanges 2 and 3. It is easy to check that $\sigma \circ \tau(1) = 2$ while $\tau \circ \sigma(1) = 3$. Thus, they are different permutations and S_n is non-abelian for $n > 2$. □

We have already written some permutations when we looked at the dihedral groups. One way to write the permutation σ is as follows.

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$$

We will improve on this notation shortly.

Example. The group S_2 consists of two elements.

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Example. We have already seen the group S_3 although we have seen it as a different group, namely, D_3. It consists of the 6 permutations,

$$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}.$$

Example. Set

$$\alpha = \begin{pmatrix}1&2&3&4&5&6\\1&4&6&5&2&3\end{pmatrix}, \beta = \begin{pmatrix}1&2&3&4&5&6\\5&2&1&4&6&3\end{pmatrix}, \gamma = \begin{pmatrix}1&2&3&4&5&6\\2&1&6&5&4&3\end{pmatrix}$$

in S_6. Determine $\alpha\beta\gamma$, β^{-1} and the order of γ.

Proof. $\alpha\beta\gamma = \begin{pmatrix}1&2&3&4&5&6\\4&2&6&3&5&1\end{pmatrix}, \beta^{-1} = \begin{pmatrix}1&2&3&4&5&6\\3&2&6&4&1&5\end{pmatrix}$ and $o(\gamma) = 2$. □

We will now try and simplify the notation of permutations allowing arguments to be more succinct. We will first need the idea of a cycle, this is a special type of permutation.

Definition 2.3.3. A permutation σ is a *cycle* if there are distinct elements $a_1, \ldots, a_k$ such that

$$\sigma(a_i) = a_{i+1} \text{ for } 1 \leqslant i < k \qquad \sigma(a_k) = a_1$$

and

$$\sigma(x) = x \text{ for } x \notin \{a_1, \ldots, a_k\}.$$

The *length* of such a cycle is k and we call σ a *k-cycle*.

To denote the k-cycle we write $(a_1\, a_2\, \ldots\, a_k)$. Note that this notation is not unique, we could also write $(a_2\, a_3\, \ldots\, a_k\, a_1)$, in fact, there are k ways to write this cycle.

Two cycles $(a_1\, \ldots\, a_k)$ and $(b_1\, \ldots\, b_l)$ are disjoint if $a_i \neq b_j$ for all i, j.

Proposition 2.3.4. *Disjoint cycles commute.*

Proof. Let α be the cycle $(a_1\, a_2\, \ldots\, a_k)$ and β be the cycle $(b_1\, b_2\, \ldots\, b_l)$. Let us now consider $\beta(\alpha(i))$. There are three possibilities.

1. $i \in \{a_1, a_2, \ldots, a_k\}$ suppose $i = a_j$. In this case $\alpha(i) = \alpha(a_j) = a_{j+1}$ or a_1 of $j = k$. Now consider $\beta(a_{j+1})$. Since these are disjoint permutations we see that $\beta(a_{j+1}) = a_{j+1}$. The reasoning is similar for a_1.

2. $i \in \{b_1, b_2, \ldots, b_l\}$. Suppose the $i = b_m$. Since the permutations are disjoint we see that $\alpha(i) = \alpha(b_m) = b_m$. Now we compute $\beta(b_m) = b_{m+1}$ or b_1 depending on whether $m = l$ or not.

3. Finally assume $i \notin \{a_1, \ldots, a_k, b_1, \ldots, b_l\}$. In this case $\alpha(i) = i$ and $\beta(i) = i$. Thus $\beta(\alpha(i)) = i$.

   Now check what happens when we compose the other way. I.e. compute $\alpha(\beta(i))$. Once again there are three cases, computing as above we get the same answer.

□

Theorem 2.3.5. *Every element of S_n can be expressed as a product of disjoint cycles. Such an expression is uniquely determined up to the ordering of the cycles and the notational redundancy within each cycle.*

Proof. Let σ be an element of S_n. Choose any $i_1 \in \{1, \ldots, n\}$. We can now construct a sequence of elements of $\{1, \ldots, n\}$ starting with i_1 by repeatedly applying σ:

$$i_1,\ i_2,\ i_3,\ \ldots,$$

so that $i_j = \sigma(i_{j-1})$ for $j \geqslant 2$. This sequence must involve repetitions. Suppose that the kth term is the first one which is a repetition. In other words, i_k has already appeared earlier in the sequence but the earlier terms themselves have each appeared for the first time. If $i_k = i_j$ with $j < k$ and $j \neq 1$ then we get a contradiction because two different elements i_{j-1}, i_{k-1} map to i_k. Therefore it must be the case that $i_k = i_1$.

In this way we see that σ involves the $k-1$-cycle

$$(i_1\ \ldots\ i_{k-1}).$$

If necessary we can repeat this starting with an $i_1' \in \{1, \ldots, n\}$ which does not appear in the first cycle. We now obtain a second cycle and it must be disjoint from the first because σ is a permutation. Continuing in this way we obtain a disjoint cycle representation of σ. □

For brevity, we will remove cycles of length 1.

Example. We can now write α, β, γ from above as

$$\alpha = (2\,4\,5\,2)(3\,6), \beta = (1\,5\,6\,3), \gamma = (1\,2)(3\,6)(4\,5).$$

Definition 2.3.6. As a consequence of the above theorem, the lengths of the various cycles of a permutation are well defined. This is known as the *cycle type* of the permutation.

Proposition 2.3.7. *Let $\sigma = \rho_1\rho_2 \ldots \rho_k$ be an expression for σ as a product of disjoint cycles of length $l_1, \ldots, l_k$. Then the order of σ is $lcm(l_1, \ldots, l_k)$.*

Proof. Since disjoint cycles commute we see that $\sigma^n = \rho_1^n\rho_2^n \ldots \rho_k^n$. So for this expression to be the trivial element we see that ρ_i^n must be trivial for all i. The order of an l cycle is l. Thus ρ_i^n is trivial if and only if l_i divides n. Thus σ^n is trivial if and only if l_i divides n for all n. The smallest such positive number is the lowest common multiple of $(l_1, \ldots, l_k)$. □

Proposition 2.3.8. *Let $k \leqslant n$. Then there are*

$$\frac{n!}{(n-k)!k}$$

cycles of length k in S_n.

Proof. There are n choices for the first element, then $n-1$ choices for the second element and so on. This gives $\frac{n!}{(n-k)!}$. However since the order only matters up to cyclic permutation we can see that we have over counted by k. Thus we obtain that the number of k-cycles is

$$\frac{n!}{(n-k)!k}.$$

□

Example. How many permutations are there of each cycle type in S_6?

Proof.

Cycle type	Number of permutations
$(1,1,1,1,1,1)$	1
$(2,1,1,1,1)$	15
$(2,2,1,1)$	45
$(2,2,2)$	15
$(3,1,1,1)$	40
$(3,2,1)$	120
$(3,3)$	40
$(4,1,1)$	90
$(4,2)$	90
$(5,1)$	144
(6)	120

□

If you forgot the numbers in the permutation and just remembered the cycle type. This ideas is captured by conjugacy in the symmetric group.

Definition 2.3.9. Two permutations σ, τ are *conjugate* if there is a permutation ρ such that $\tau = \rho\sigma\rho^{-1}$. We say that ρ *conjugates* σ to τ.

Lemma 2.3.10. *Any two k-cycles are conjugate. In fact, given the cycle $\tau = (a_1\, a_2\, \ldots\, a_k)$ we can show that $\sigma\tau\sigma^{-1} = (\sigma(a_1)\, \sigma(a_2)\, \ldots\, \sigma(a_k))$.*

Proof. Exercise □

Theorem 2.3.11. *Two permutations are conjugate if and only if they have the same cycle type.*

Proof. Let $\sigma = \sigma_1 \ldots \sigma_k$ where σ_i and σ_j are disjoint. Assume that τ is conjugate to σ, so there is a ρ such that $\rho\sigma\rho^{-1} = \tau$. Then $\rho\sigma\rho^{-1} = \rho\sigma_1 \ldots \sigma_k\rho^{-1} = \rho\sigma_1\rho^{-1}\rho\sigma_2\rho^{-1} \ldots \rho\sigma_k\rho^{-1}$. By the above lemma $\rho\sigma_1\rho^{-1}$ is a cycle with the same length as σ_1. Also we can see that $\rho\sigma_i\rho^{-1}$ is disjoint from $\rho\sigma_j\rho^{-1}$ so we have written τ as a product of disjoint cycles with the same lengths as those for σ. Thus they have the same cycle type.

Let σ and τ be permutations with the same cycle type. First write σ as a product of disjoint cycles $\sigma = \sigma_1 \ldots \sigma_k$ and τ as a product of disjoint cycles $\tau = \tau_1 \ldots \tau_k$, where σ_i and τ_i have the same length. Define ρ as the permutation which sends that j-th element of σ_i to the j-th element of τ_i. The lemma above shows that $\rho\sigma\rho^{-1} = \tau$. □

Example. Find a permutation that conjugates $(1\,2)(3\,4)(5\,6\,7\,8)$ to $(5\,8)(2\,7)(1\,6\,4\,3)$.

Proof. The permutations $(1\,5)(2\,8\,3)(4\,7)(6)$ is a permutation that has the desired effect. □

Definition 2.3.12. A *transposition* is another name for a 2-cycle.

A permutation is said to be *odd* (respectively *even*) if it is a product of an odd (respectively even) number of transpositions.

Theorem 2.3.13. *Every permutation can be written as a product of transpositions.*

Proof. Since every permutation is a product of disjoint cycles, it is enough to show that each cycle can be written as a product of transpositions. This is certainly the case as $(a_1\,a_2\,\ldots\,a_k) = (a_1\,a_2)(a_2\,a_3)\ldots(a_{k-1}\,a_k)$. □

Theorem 2.3.14. *There is no permutation which is both even and odd.*

Proof. content... □

Remark 2. Note that cycles of even length are odd permutations and cycles of odd length are even permutations.

The above is somewhat annoying but parity has similar properties to additions of odd and even numbers.

Proposition 2.3.15. *Let $\sigma, \tau \in S_n$. Then*

- *If σ, τ are even, then $\sigma\tau$ is even.*
- *if σ, τ are odd, then $\sigma\tau$ is even.*
- *if σ is odd and τ is even, then $\sigma\tau$ is odd.*
- *if σ is even and τ is odd, then $\sigma\tau$ is odd.*

Proof. This follows from the definitions of even and odd. □

Proposition 2.3.16. *The set of even permutations form a group, called the* alternating group*, denoted A_n.*
For $n \geqslant 2$ the order of A_n is $\frac{n!}{2}$.
For $n \geqslant 4$ A_n is non-abelian.

2.4 Subgroups

Definition 2.4.1. Let G be a group. A subset $H \subset G$ is a *subgroup* if the group operation $*$ restricts to H to make H a group. That is H is a subgroup of G if:

1. $e \in H$,
2. if $g, h \in H$, then $gh \in H$,
3. if $g \in H$, then $g^{-1} \in H$.

If H is a subgroup of G, then we write $H \leqslant G$.

Example. A_n is a subgroup of S_n.

Proposition 2.4.2. *Let G be a group. Let H be a subset. Then H is a subgroup if and only if H is non-empty and whenever $g, h \in H$, then $gh^{-1} \in H$.*

Proof. We must check the three conditions of the above definition.

Firstly since $H \neq \varnothing$ there is a $a \in H$. Setting $g = a = h$ we see that $aa^{-1} = e \in H$.

We now move onto the third point. Let $a \in H$ we have already shown that $e \in H$. So taking $g = e$ and $h = a$ we see that $ea^{-1} = a^{-1} \in H$.

Now that we know that $a^{-1} \in H$ for all $a \in H$. Given $a, b \in H$ let $g = a$ and $h = b^{-1}$. Then $a(b^{-1})^{-1} = ab \in H$.

Thus H satisfies the three axioms of being a subgroup. □

Example. The subgroups of S_3 are

$$\{e\}, \{e, (1\,2)\}, \{e, (2\,3)\}, \{e, (1\,3)\}, \{e, (1\,2\,3), (3\,2\,1)\} = A_3, S_3.$$

Example. The subgroups of D_4 are

$$\{e\}, \{e, r^2\}, \{e, s\}, \{e, rs\}, \{e, r^2s\}, \{e, r^3s\},$$

$$\{e, r, r^2, r^3\}, \{e, rs, r^2, r^3s\}, \{e, s, r^2, r^2s\}, D_4.$$

Example. The subgroups of C_6 are

$$\{e\}, \{e, g^3\}, \{e, g^2, g^4\}, C_6.$$

The only subgroups of C_5 are $\{e\}$ and C_5.

Proposition 2.4.3. *Let G be a group and H, K subgroups of G. Then $H \cap K$ is a subgroup of G.*

Proof. Exercise □

Definition 2.4.4. The *subgroup generated by a set S* , denoted $\langle S \rangle$, is the smallest subgroup containing S. I.e. $\langle S \rangle = \bigcap_{S \subset H} H$ where H is a subgroup of G.

If $g \in G$, then we write $\langle g \rangle$ rather than the more cumbersome $\langle \{g\} \rangle$.

If $\langle S \rangle = G$, then the set S is called a *generating set* for G.

Theorem 2.4.5. *The subgroup $\langle S \rangle$ is equal to the set*

$$H = \{w \in G \mid w = s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_k^{\epsilon_k} \text{ where } s_i \in S, \epsilon_i \in \{-1, 0, 1\}\}.$$

Proof. Since any subgroup is closed under products and inversion. Since subgroups are closed under taking products and inverses we see that H is contained in any subgroup containing S. Thus $H \subset \langle S \rangle$.

To see the other direction we must show that H is a subgroup. H contains the identity by setting $k = 0$.

We now check that H is closed under taking inverses. To see this let $w \in H$, then $w = s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_k^{\epsilon_k}$ and $w^{-1} = s_k^{-\epsilon_1} s_{k-1}^{-\epsilon_2} \dots s_1^{-\epsilon_k}$ which is also an element of H.

Finally, we must check that it is closed under taking products. Let $w, v \in H$. Then $w = s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_k^{\epsilon_k}$ and $v = t_1^{\delta_1} t_2^{\delta_2} \dots t_k^{\delta_l}$ where $s_i, t_i \in S$ and $\epsilon_i, \delta_i \in \{-1, 0, 1\}$. Then $wv = s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_k^{\epsilon_k} t_1^{\delta_1} t_2^{\delta_2} \dots t_k^{\delta_l}$ which is also an element of H.

Thus $\langle S \rangle \subset H$ and the proof is complete. □

Example. Determine $\langle S \rangle$ in each of the following cases:

1. $G = \mathbb{Z}$ and $S = \{16, 56\}$.
2. $G = S_5$ and $S = \{(1\,2)(3\,4), (1\,3)(2\,4)\}$.

Example. Show that if G is abelian and $g, h \in G$, then $\langle g, h \rangle = \{g^n h^m \mid n, m \in \mathbb{Z}\}$.

Proof. By the above Theorem we can see that $\langle g, h \rangle = \{g^{m_1} h^{n_1} g^{m_2} \dots h^{n_k}\}$. Since the group is Abelian we know that $gh = hg$. Thus we can replace this word by a word where all the g's are on the left. We obtain that $\{g^{m_1} h^{n_1} g^{m_2} \dots h^{n_k} = g^{m_1 + \dots + m_k} h^{n_1 + \dots + n_k}$. □

Proposition 2.4.6. *Let G be a group and $g \in G$. Then $\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\}$.*

Proof. The proof is the same as above, noting that we can combine powers of g. □

Definition 2.4.7. The *order* of an element $g \in G$, denoted $o(g)$, is the smallest integer $n > 0$ such that $g^n = e$. If no such integer exists, then $o(g) = \infty$.

Proposition 2.4.8. *If $o(g)$ is finite, then $\langle g \rangle = \{e, g, \dots, g^{o(g)-1}\}$.*

Proof. We can always replace g^k with a representative from this set by adding or subtracting $o(g)$. Thus we can see that $\langle g \rangle \subset \{e, g, \dots, g^{o(g)-1}\}$. This completes the proof since the other inclusion is clear. □

We can now redefine what it means to be a cyclic group. G is cyclic if and only if there exists a $g \in G$, such that $G = \langle g \rangle$.

Remark 3. Note that in a finite group $G = \langle g \rangle$ if and only if $o(g) = |G|$.

Theorem 2.4.9. *Let G be a cyclic group and H a subgroup of G. Then H is cyclic.*

Proof. If $H = \{e\}$, then H is cyclic and generated by e. Suppose that $H \neq \{e\}$.

Let g be a generator for G. So $G = \langle g \rangle$. Define $l = \min\{n \mid n > 0 g^n \in H\}$. Since H contains an element other than e. We see that $g^m \in H$ for some m. Thus g^{-m} is also in H and we can see that l is well-defined.

We will now show that $H = \langle g^l \rangle$. Since $g^l \in H$ we see that $\langle g^l \rangle \subset H$. Suppose that $g^m \in H$. Then we see that $g^{m+l} = g^m g^l \in H$ and $g^{m-l} = g^m (g^l)^{-1} \in H$. So we can find an a such that $a = m + bl$, $0 \leqslant a < l$ and $g^a \in H$. By minimality of l we see that $a = 0$ and $m = -bl$. Thus this element was in $\langle g^l \rangle$, this completes the proof. □

Proposition 2.4.10. *Let m, n be non-zero integers. By Theorem 2.4.9 we have that*

$$\langle m, n \rangle = \langle h \rangle \qquad \langle m \rangle \cap \langle n \rangle = \langle l \rangle$$

for some $h, l \in \mathbb{Z}$. Then h and l have the following properties:

1. *$h \mid n$ and $h \mid m$,*
2. *if $x \mid n$ and $x \mid m$, then $x \mid h$*
3. *there exists $u, v \in \mathbb{Z}$ such that $un + vm = h$.*
4. *$m \mid l$ and $n \mid l$,*
5. *if $m \mid x$ and $n \mid x$, then $l \mid x$.*

Proof. We can rephrase the property that i divides j by $j \in \langle i \rangle$.

The first property follows since $\langle m \rangle \subset \langle m, n \rangle$. So $m \in \langle h \rangle$. Similarly for n.

The third part follows since $h \in \langle m, n \rangle$ and so $h = un + vm$ by definition of $\langle m, n \rangle$.

The second part now follows. Assume $m = ax$ and $n = bx$. Then $h = un + vm = ubx + vax = x(ub + va)$ and the result follows.

For the fourth part, since $l \in \langle m \rangle \cap \langle n \rangle$ we see that $\langle m \rangle$ and $l \in \langle n \rangle$.

Finally if $x \in \langle m \rangle$ and $x \in \langle n \rangle$, then $x \in \langle m \rangle \cap \langle n \rangle = \langle h \rangle$. This completes the proof. □

Definition 2.4.11. By definition h is the *greatest common divisor* and l is the *lowest common multiple*.

2.5 Lagrange's theorem

Recall that the order of an element g is the minimal $n > 0$ such that $g^n = e$. If no such n exists, then we say g is of infinite order.

Definition 2.5.1. Let H be a subgroup of a group G.

Then the *left cosets* of H in G are the sets

$$gH = \{gh \mid h \in H\}.$$

The *right cosets* of H in G are the sets

$$Hg = \{hg \mid h \in H\}.$$

Notation 2.5.2. We write G/H for the set of left cosets of H. The cardinality of G/H is the *index* of H in G.

Note that different elements g_1, g_2 can represent the same left coset. I.e. $g_1H = g_2H$ yet $g_1 \neq g_2$.

In general, $gH \neq Hg$, there are certain special cases where this will be true.

Example. Let $G = S_3$ and $H = \{e, (1\,2)\}$. Then

$$\begin{aligned} eH = (1\,2)H = \{e, (1\,2)\}, \qquad & He = H(1\,2) = \{e, (1\,2)\}, \\ (1\,3)H = (1\,3\,2)H = (1\,3), (1\,3\,2), \qquad & H(1\,3) = H(1\,2\,3) = (1\,3), (1\,2\,3), \\ (2\,3)H = (1\,2\,3)H = (2\,3), (1\,2\,3), \qquad & H(2\,3) = H(1\,3\,2) = (2\,3), (1\,3\,2). \end{aligned}$$

Lemma 2.5.3. *Let $H \leqslant G$ and $g, k \in G$. Then $gH = kH$ if and only if $g^{-1}k \in H$. Similarly $Hg = Hk$ if and only if $kg^{-1} \in H$.*

Proof. Suppose that $gH = kH$. Since $k \in kH = gH$ we see that $k = gh$ for some $h \in H$. Thus $h = g^{-1}k$ and so $g^{-1}k \in H$.

Suppose now that $g^{-1}k \in H$ note that $k^{-1}g \in H$ as well. Let $a \in gH$. Then $a = gh$ for some $h \in H$. We have the following chain of equalities.

$$a = gh = g(g^{-1}k)(k^{-1}g)h = k(k^{-1}g)h.$$

Since $k^{-1}g \in H$, we see that this is of the form kh' where $h' = (k^{-1}g)h \in H$. Thus $gH \subset kH$. The proof of the other inclusion is the same.

The statement about right cosets is left as an exercise. □

Remark 4. This allows us to put an equivalence relation on G by making $g \sim h$ if and only if $g^{-1}h \in H$. The equivalence classes of this relation are the left cosets of H.

From the examples that have been given so far you may have noticed that in the case where G is finite and $H \leqslant G$, we have that $|H| \mid |G|$. This is not a coincidence! The following theorem is known as Lagrange's theorem.

Theorem 2.5.4. *Let G be a finite group and $H \leqslant G$. Then the order of H divides the order of G.*

Proof. By the above remark we see that the cosets are equivalence classes of an equivalence relation. Thus, the cosets partition the set G. We must now prove that $|H| = |gH|$ for all $g \in G$. The map $\varphi\colon H \to gH$ given by $\varphi(h) = gh$ is a bijection. Indeed it has inverse $\psi\colon gH \to H$, given by $\psi(gh) = h$.

Since we have a partition we can see that $|G| = \sum_{gH \in G/H} |gH| = \sum_{gH \in G/H} |H| = |G/H||H|$. This completes the proof. □

The converse of Lagrange's theorem is false in general. However, we will see some partial converses to Lagrange's theorem.

Example. Find all the subgroups of C_{31}, D_5 and $C_5 \times C_5$.

Proposition 2.5.5. *Let G be a finite group. Then $o(g)$ divides $|G|$ for all $g \in G$.*

Proof. We know that $|\langle g \rangle| = o(g)$. We also know that $\langle g \rangle$ is a subgroup, thus the previous theorem completes the proof. □

Corollary 2.5.6. *Let g be an element of a finite group. Then $g^{|G|} = e$.*

Proof. Since $g^k = e$ if and only if $o(g)$ divides k. The above theorem shows that $g^{|G|} = e$. □

The converse of this is obviously false. Otherwise this would imply that every finite group is cyclic.

Corollary 2.5.7. *Let G be a group such that $|G| = p$ where p is prime. Then G is cyclic.*

Proof. In a group there is only one element of order 1, namely e. If $|G| = p$, then any element has order 1 or p. Thus we can find an element of order p. This element is thus a generator for G. □

Definition 2.5.8. Let $(\mathbb{Z}/n\mathbb{Z})^\times = \{[a] \mid \gcd(a, n) = 1\}$. This is called the *group of units of* $\mathbb{Z}/n\mathbb{Z}$. This is a group with operations $[a] * [b] = [ab]$.

It is clear that this is associative, the identity is $[1]$ and by proposition 1.5.2 we have inverses.

The following two theorems are very useful in cryptography. The first is known as Fermat's little theorem and was proved by Fermat in 1640.

Theorem 2.5.9. *Let p be a prime number and $a \in \mathbb{Z}$ be an integer such that p does not divide a. Then*

$$a^{p-1} = 1 \mod p.$$

Proof. Since the group $(\mathbb{Z}/p\mathbb{Z})^\times$ has order $p - 1$. Thus for any element g we have that $g^{p-1} = [1]$. □

The second is an extension of this proved by Euler. It requires Euler's phi function which counts the number of integers coprime to n.

Definition 2.5.10. Euler's φ-function is defined to be the number of integers $0 < i < n$ such that i and n are coprime.

The following allows one to calculate $\varphi(n)$ we will not prove it here.

Theorem 2.5.11. *Euler's φ-function satisfies the following three properties.*

- *For prime numbers p, $\varphi(p) = p - 1$,*
- *For prime numbers p, $\varphi(p^k) = p^k - p^{k-1}$,*
- *If n, m are coprime integers, then $\varphi(mn) = \varphi(m)\varphi(n)$.*

We are now ready for Euler's theorem from 1736.

Theorem 2.5.12. *Let a, n be integers such that a and n are coprime. Then*

$$a^{\varphi(n)} = 1 \mod n.$$

Proof. The key point here is that the order of $(\mathbb{Z}/n\mathbb{Z})^\times$ is $\varphi(n)$. □

The key point of this section is see the power of defining an equivalence relation on a group. We can define another equivalence relation by xRy if $x = y$ or $x = y^{-1}$. This allows us to prove the following two theorems.

Corollary 2.5.13. *If p is a prime, then $(p-1)! = -1 \mod p$.*

Proof. Consider the elements of $(\mathbb{Z}/p\mathbb{Z})^\times$ which satisfy the identity $[x]^2 = [1]$. This is the same as saying that $[x-1][x+1] = [x^2-1] = [0]$. We have seen that the product of two non-zero elements is non-zero. This this equality tells us that $[x-1] = [0]$ or $[x+1] = [0]$. Thus $[x] = [1]$ or $[x] = [-1]$. Thus in the product every element pairs with an inverse except for $[1]$ and $[-1]$ and we conclude that $[(p-1)!] = [-1]$. □

Corollary 2.5.14. *Let G be a group of even order. Then there is an element of order 2.*

Proof. Exercise □

2.6 Homomorphisms and isomorphisms

In linear algebra we have a natural notion of map between vector spaces, namely that of a linear map. In group theory we have the following,

Definition 2.6.1. Let $(G, *_G)$ and $(H, *_H)$ be groups. A map $\varphi\colon G \to H$ is called a *homomorphism* if $\varphi(x *_G y) = \varphi(x) *_H \varphi(y)$ for all $x, y \in G$.

A bijective homomorphism is called an *isomorphism*. If there is an isomorphism $\varphi\colon G \to H$, then we write $G \cong H$.

Much like vector spaces isomorphism gives an equivalence relation on the class of groups.

Proposition 2.6.2. *Let G be a cyclic group. If G is finite, then there is an n such that $G \cong \mathbb{Z}/n\mathbb{Z}$. If G is infinite, then $G \cong \mathbb{Z}$.*

Proof. Suppose that $|G| = n$ and G is cyclic generated by g. So $G = \{e, g, g^2, \dots g^{n-1}\}$. Define a map $\varphi\colon G \to \mathbb{Z}/n\mathbb{Z}$ given by $\varphi(g^k) = [k]$. This is an isomorphism.

Suppose that G is infinite and generated by g. Then define a function $\varphi\colon G \to \mathbb{Z}$ given by $\varphi(g^k) = k$. This is an isomorphism. □

Definition 2.6.3. Two integers m, n are coprime, if $\gcd(m, n) = 1$.

Theorem 2.6.4. *Let m, n be coprime integers. Then $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}/mn\mathbb{Z}$.*

Proof. Consider the order of the element $([1], [1])$. The k-th power of this element is $([k], [k])$ which is the trivial element if and only if m and n both divide k. Thus the order of $([1], [1])$ is the lowest common multiple of m and n. If m and n are coprime, then the lowest common multiple is mn. Thus the group is cyclic as $o([1], [1]) = |\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}|$. □

With other theorems we have seen we can now classify all groups of order up to 7.

Theorem 2.6.5. *Up to isomorphism, the groups of order $\leqslant 7$ are:*

- *Order 1:* $\{e\}$.
- *Order 2:* $\mathbb{Z}/2\mathbb{Z}$.
- *Order 3:* $\mathbb{Z}/3\mathbb{Z}$.
- *Order 4:* $\mathbb{Z}/4\mathbb{Z}$ *or* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- *Order 5:* $\mathbb{Z}/5\mathbb{Z}$.
- *Order 6:* $\mathbb{Z}/6\mathbb{Z}$ *or* D_3.
- *Order 7:* $\mathbb{Z}/7\mathbb{Z}$.

This situation becomes much more complicated for larger orders. There are already 5 groups of order 8. We have seen 4 of these groups: $\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and D_4. The last group is Q_8.

Definition 2.6.6. The quarternion group Q_8 is the set $\{\pm 1, \pm i, \pm j, \pm k\}$ with the multiplication defined by $i^2 = j^2 = k^2 = -1$ and $ij = k, jk = i, ki = j, ijk = -1$.

Definition 2.6.7. An *automorphism* is an isomorphism from G to G.
A *endomorphism* is a homomorphism from G to G.
An injective homomorphism is called a *monomorphism*.
A surjective homomorphism is called a *epimorphism*.

It is easy to check that the composition of two homomorphisms is again a homomorphism. This also shows that the composition of two isomorphisms is and isomorphism. Thus much like the symmetric groups we can form a group of automorphisms.

Definition 2.6.8. Let G be a group. We define the *automorphism group of* G to be $Aut(G) = \{\varphi \colon G \to G \mid \varphi \text{ is an isomorphism}\}$. This is a group with composition as the group operation.

Lemma 2.6.9. *Let $\varphi \colon G \to H$ be a homomorphism. Then the following hold:*

- $\varphi(e_G) = e_H$.
- $\varphi(g^n) = (\varphi(g))^n$.
- $\varphi(g^{-1}) = (\varphi(g))^{-1}$

Proof. Exercise □

Proposition 2.6.10. *Let G and H be group and $\varphi \colon G \to H$ be a homomorpshism. Then $o(\varphi(g))$ divides $o(g)$ for all $g \in G$. Moreover if φ is an isomorphim, then $o(\varphi(g)) = o(g)$.*

Proof. Removed for grading purposes. □

Example. The map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ given by k goes to $[k]$ is a homomorphism.

Example. If H is a subgroup of G, then the inclusion map $i\colon H \to G$ given by $i(h) = h$ is a homomorphism.

Example. For any groups G and H. The map $\varphi\colon G \to H$ given by $\varphi(g) = e_H$ is a homomorphism. This is called the trivial homomorphism.

Example. Let G and H be groups. Then the maps

$$\pi_1\colon G \times H \to G \qquad \pi_1((g,h)) = g$$

and

$$\pi_2\colon G \times H \to H \qquad \pi_2((g,h)) = h$$

are homomorphisms.

Example. The map $\det\colon S_n \to \{1,-1\}$ given by

$$\det(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

is a homomorphism.

Example. The map $\det\colon GL_n(\mathbb{R}) \to \mathbb{R}$ is a homomorphism.

Example. The maps $\operatorname{trace}\colon M_n(\mathbb{R}) \to \mathbb{R}$ is a homomorphism.

Example. The map $\log\colon (0,\infty) \to \mathbb{R}$ is a homomorphism. Moreover it has an inverse $\exp\colon \mathbb{R} \to (0,\infty)$ so it is in fact an isomorphism.

Example. The map $\varphi\colon \mathbb{R} \to \mathbb{C} \setminus \{0\}$ given by $\varphi(x) = e^{ix}$ is a homomorphism.

Example. The map $\varphi\colon \mathbb{C} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ given by $\varphi(z) = |z|$ is a homomorphism.

Definition 2.6.11. Let G be a group and $a \in G$. We can associate to a the map $\theta_a\colon G \to G$ given by $\theta_a(g) = a^{-1}ga$. This is known as *conjugating* by a.

Given $g, h \in G$ we say that g and h are *conjugate* if there is an $a \in G$ such that $h = \theta_a(g)$.

Proposition 2.6.12. *Conjugation by a is an isomorphism for every $a \in G$.*

Proof. The function is a homomorphism since $\theta_a(gh) = agha^{-1} = (aga^{-1})(aha^{-1}) = \theta_a(g)\theta_a(h)$. We can also see that it is bijective since it has an inverse, namely $\theta_{a^{-1}}$. □

Corollary 2.6.13. *If g and h are conjugate, then $o(g) = o(h)$.*

If g and h are conjugate, then g^{-1} and h^{-1} are conjugate.

Similarly to linear maps we have a notion of kernel and image. In linear algebra these are subspaces in group theory, unsurprisingly, these are subgroups.

Definition 2.6.14. Let $\varphi\colon G \to H$ be homomorphism between two groups.

The *kernel* of φ, written $\ker(\varphi)$ is

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = e_H\}.$$

The *image* of φ, written $\mathrm{Im}(\varphi)$ is

$$\mathrm{Im}(\varphi) = \{h \in H \mid \exists g \in G \text{ s.t. } \varphi(g) = h\}.$$

Proposition 2.6.15. *Let $\varphi\colon G \to H$ be a homomorphism. Then $\ker(\varphi) \leqslant G$ and $\mathrm{Im}(\varphi) \leqslant H$.*

Proof. We must check the three axioms of a subgroup in each case.

We can see that $e_H = \varphi(e_G) \in \mathrm{Im}(\varphi)$.

Given $h, h' \in \mathrm{Im}(\varphi)$ suppose that $h = \varphi(g)$ and $h' = \varphi(g')$. Then $hh' = \varphi(g)\varphi(g') = \varphi(gg') \in \mathrm{Im}(\varphi)$.

Finally given $h \in \mathrm{Im}(\varphi)$ suppose $h = \varphi(g)$, then $h^{-1} = \varphi(g)^{-1} = \varphi(g^{-1}) \in \mathrm{Im}(\varphi)$.

We proceed similarly for the kernel. Since $\varphi(e_G) = e_H$, we see that $e_G \in \ker(\varphi)$.

Suppose that $g, g' \in \ker(\varphi)$. Then $\varphi(gg') = \varphi(g)\varphi(g') = e_H e_H = e_H$, thus $gg' \in \ker(\varphi)$.

Similarly suppose $g \in \ker(\varphi)$, then $\varphi(g^{-1}) = \varphi(g)^{-1} = e_H^{-1} = e_H$. Thus $g^{-1} \in \ker(\varphi)$. □

Example. The map $\varphi\colon \mathbb{Z} \to C_n$ given by $k \mapsto g^k$ has kernel $n\mathbb{Z}$ and image C_n.

Example. The map $\det\colon S_n \to \{1, -1\}$ has kernel A_n and image $\{1, -1\}$.

Example. The maps $\det\colon GL_n(\mathbb{R}) \to \mathbb{R}$ has kernel $SL_n(\mathbb{R})$ and image $\mathbb{R}$.

We end our discussion here with a first step to proving the first isomorphism theorem.

Proposition 2.6.16. *A homomorphism $\varphi\colon G \to H$ is injective if and only if $\ker(\varphi) = \{e_G\}$.*

Proof. Suppose that φ is injective, then $\varphi(g) = e_H = \varphi(e_G)$ tells us that $g = e_G$. Thus $\ker(\varphi) = \{e_G\}$.

Assume that $\ker(\varphi) = \{e_G\}$. Suppose that $\varphi(x) = \varphi(y)$ then we have the following sequence of equivalences:

$$\begin{aligned}
& \varphi(x) = \varphi(y) \\
\Leftrightarrow \quad & \varphi(x)^{-1}\varphi(y) = e_H \\
\Leftrightarrow \quad & x^{-1}y \in \ker(\varphi) \\
\Leftrightarrow \quad & x^{-1}y = e_G \\
\Leftrightarrow \quad & x = y.
\end{aligned}$$

Thus we see that φ is injective. □

The above proof actually showed that if $\varphi(x) = \varphi(y)$, then $x^{-1}y \in \ker(\varphi)$. Thus we have the following corollary.

Corollary 2.6.17. *A homomorphism is constant on cosets of* $\ker(\varphi)$ *and takes different values on different cosets.*

2.7 Normal subgroups and quotients

In this section we will talk about quotient groups. These form some of the key ideas in group theory. They have the downside of begin very abstract, so don't worry if this isn't familiar first time round.

Recall, that if G is a group and $H \leqslant G$, then G/H is the set of left cosets of H in G.

Definition 2.7.1. Let H be a subgroup of G. Then H is a *normal* subgroup, denoted $H \lhd G$, if for all $g \in G$ we have

$$Hg = gH.$$

One can say that H is normal if left cosets and right cosets agree.

WARNING: This does not mean that $hg = gh$ for all $g \in G$ and $h \in H$.

Proposition 2.7.2. *The following are equivalent:*

1. H *is normal in* G.
2. $ghg^{-1} \in H$ *for all* $h \in H$ *and all* $g \in G$.
3. $g^{-1}hg \in H$ *for all* $h \in H$ *and all* $g \in G$.

Proof. The last two conditions are equivalent since the map $g \mapsto g^{-1}$ is a bijection.

We will show that the first and second are equivalent.

To see that the first implies the second. Let $h \in H$ and $g \in G$. Since $gH = Hg$, we see that $gh = h'g$ for some $h' \in H$. Thus $ghg^{-1} = h' \in H$. This proves the second statement.

Suppose that $ghg^{-1} \in H$ for all $h \in H, g \in G$. Then $ghg^{-1} = h'$ and so $gh = h'g$. Thus $gH \subset Hg$. Switching the roles of h and h' we see the other inclusion. □

There are always 2 normal subgroups of any group G. These are G and $\{e\}$. If there are no other normal subgroups, then G is called *simple.*

Proposition 2.7.3. *If* H *is a subgroup of* G *and* $|G/H| = 2$*, then* H *is a normal subgroup.*

Proof. Exercise □

Definition 2.7.4. Let G be a group. Then the *centre* of G, denoted $Z(G)$, is the set

$$Z(G) = \{g \in G \mid hg = gh \text{ for all } h \in G\}.$$

Proposition 2.7.5. *Let G be a group. Then $Z(G) \lhd G$.*

Proof. It was proved on the midterm that $Z(G)$ is a subgroup of G. To see that it is normal note that $hgh^{-1} = g$ for all $g \in Z(G)$ so it is certainly normal. □

Proposition 2.7.6. *Let $\varphi\colon G \to H$ be a homomorphism. Then* $\ker(phi) \lhd G$.

Proof. To see that the kernel is a normal subgroup. Let $k \in \ker(\varphi)$ and $g \in G$. Consider $\varphi(gkg^{-1}) = \varphi(g)\varphi(k)\varphi(g)^{-1} = \varphi(g)e\varphi(g)^{-1} = e$. Thus $gkg^{-1} \in \ker(\varphi)$ and so $\ker(\varphi)$ is normal. □

One could try and define a binary operation on the set G/H by defining $g_1H * g_2H = g_1g_2H$. However one must check that it does not matter which choice of g_1 and g_2 are used. This is the key reason for defining normal subgroups.

Proposition 2.7.7. *Let $H \leqslant G$. Then the binary operation $g_1H * g_2H = g_1g_2H$ is well defined if and only if H is normal.*

If H is a normal subgroup of G. Then $(G/H, *)$ is a group.

Proof. Suppose that H is normal in G. Let $g_1H = k_1H$ and $g_2H = k_2H$. To show that $*$ is well defined we must show that $g_1g_2H = k_1k_2H$. This is equivalent to the saying that $g_2^{-1}g_1^{-1}k_1k_2 \in H$. Since $g_2H = k_2H$ we see that $g_2 = k_2h$ for some $h \in H$. Thus $g_2^{-1}g_1^{-1}k_1k_2 \in H$ is equivalent ot $g_2^{-1}g_1^{-1}k_1k_2h \in H$ which is the same as $g_2^{-1}g_1^{-1}k_1g_2 \in H$. Since $k_1H = g_1H$ we see that $g_1^{-1}k_1 \in H$. Thus since H is normal we see that $g_2^{-1}g_1^{-1}k_1g_2 \in H$ and $*$ is well defined.

Suppose that $*$ is well defined. Then since $hH = H$ for all $h \in H$. We see that $hH * gH = hgH = gH = H * gH$ for all $g \in G$. This is the same as $g^{-1}hg \in H$ which is equivalent to H being normal.

To see that $(G/H, *)$ is a group we check the three axioms. The operation $*$ is associative since the operation in G is associative. The identity is the coset H. The inverse of gH is the coset $g^{-1}H$. □

Definition 2.7.8. If $H \lhd G$, then $(G/H, *)$ is the *quotient group.*

Proposition 2.7.9. *Let G be a group and $H \leqslant G$. Then $H\lhd$ if and only if it is the kernel of some homomorphism.*

Proof. The function $\varphi\colon G \to G/H$ given by $\varphi(g) = gH$ is a homomorphism. The kernel is exactly $\{g \in G \mid \varphi(g) = H\}$, this is the same as $\{g \in G \mid gH = H\} = \{g \in G \mid g \in H\} = H$. Thus H is the kernel of a homomorphism. □

Example. Let $G = \mathbb{Z}$ and $H = n\mathbb{Z}$. H is a normal subgroup of G. We can see that the quotient $G/H = \{n\mathbb{Z}, 1 + n\mathbb{Z}, \ldots, (n-1) + n\mathbb{Z}\}$. This can naturally be identified with the integers modulo n, or the cyclic group of order n.

Example. Let $G = S_n$ and $H = A_n$. Then $G/H = \{A_n, (1\,2)A_n\} \cong C_2$.

Example. Let $G = \mathbb{C}^*$ and $H = \{z \in \mathbb{C} \mid |z| = 1\}$. Then $G/H \cong (0, \infty)$. Essentially we have forgotten the argument of the complex number and just remembered the modulus.

Example. Let $G = AGL_n(\mathbb{R})$, denote the group of affine linear transformations. I.e. functions $\mathbb{R}^n \to \mathbb{R}^n$ which are of the form $x \mapsto Ax + b$ where $A \in GL_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Let T be the subgroup consisting of translations, that is the subgroup where $A = \mathrm{id}$.

T is a normal subgroup of G. And $G/T \cong GL_n(\mathbb{R})$.

2.8 Isomorphism theorems

Understanding quotient groups can be hard, especially understanding them in the abstract. To make this study easier, we have the first isomorphism theorem. This allows us to identify a quotient with a certain image in another group.

Theorem 2.8.1. *Let G and H be groups and $\varphi\colon G \to H$ be a homomorphism. Then $G/\ker(\varphi) \cong \mathrm{Im}(\varphi)$.*

Proof. We define a function $\Phi\colon G/\ker(\varphi) \to \mathrm{Im}(\varphi)$ by $\Phi(g\ker(\varphi)) = \varphi(g)$. We must prove four things, firstly that the map Φ is well defined, secondly Φ is a homomorphism, thirdly Φ is injective, fourthly Φ is surjective.

To see that Φ is well defined we must check that if $g\ker(\varphi) = h\ker(\varphi)$, then $\varphi(g) = \varphi(h)$.

$$\begin{aligned} & g\ker(\varphi) = h\ker(\varphi) \\ \Rightarrow \quad & g^{-1}h \in \ker(\varphi) \\ \Rightarrow \quad & \varphi(g^{-1}h) = e \\ \Rightarrow \quad & \varphi(g) = \varphi(h) \end{aligned}$$

Thus we see that the function is well defined.

To see that it is a homomorphism we see that $\Phi(gh\ker(\varphi)) = \varphi(gh) = \varphi(g)\varphi(h) = \Phi(g\ker(\varphi))\Phi(h\ker(\varphi))$.

We must now check that it is injective. Suppose that $\Phi(g\ker(\varphi)) = \Phi(h\ker(\varphi))$, then we get the following chain of implications.

$$\begin{aligned} & \Phi(g\ker(\varphi)) = \Phi(h\ker(\varphi)) \\ \Rightarrow \quad & \varphi(g) = \varphi(h) \\ \Rightarrow \quad & \varphi(g^{-1}h) = e \\ \Rightarrow \quad & g^{-1}h \in \ker(\varphi) \\ \Rightarrow \quad & g\ker(\varphi) = h\ker(\varphi) \end{aligned}$$

Thus Φ is injective.

Finally it is clear it is surjective since given $h \in \mathrm{Im}(\varphi)$ we see that $h = \varphi(g)$ for some $g \in G$ and so $\Phi(g\ker(\varphi)) = \varphi(g) = h$. □

This theorem can sometimes allow us to understand quotient groups via instead understanding subgroups which are the image of a homomorphism.

Corollary 2.8.2. *Let $\varphi\colon G \to H$ be a homomorphism. Assume G is finite. Then $|G| = |\mathrm{Im}(\varphi)| \times |\ker(\varphi)|$.*

Example. For $\det\colon S_n \to \{-1, 1\}$, the above reads that $S_n/A_n \cong C_2$.

Example. For $\det\colon GL_n(\mathbb{R}) \to \mathbb{R}^*$. The isomorphism theorem reads

$$GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^*.$$

Example. For the projection onto the first coordinate $G_1 \times G_2 \to G_1$. The isomorphism theorem reads that $(G_1 \times G_2)/(\{e\} \times G_2) \cong G_1$.

Example. For $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(x) = nx$, the isomorphism theorem reads that $\mathbb{Z} \cong \mathbb{Z}/\{0\} \cong n\mathbb{Z}$.

Example. For $\varphi\colon \mathbb{Z} \to C_n$ given by $\varphi(k) = g^k$, the isomorphism theorem tells us that $\mathbb{Z}/n\mathbb{Z} \cong C_n$.

3 Automorphism groups

Let G be a group. Then we can consider the set $Aut(G) = \{\varphi\colon G \to G \mid \varphi \text{ is an isomorphism}\}$.

Theorem 3.0.1. *$Aut(G)$ is a group where the binary operation is composition.*

Proof. This operation is associative, since composition of functions is associative.

The identity element is the function id_G.

Since each isomorphism is a bijection we see that there are inverses which are also bijections. One can also see that the inverse will satisfy the property of being a homomorphism. □

We call $Aut(G)$ the *automorphism group of G*.

Example. For a natural number n the group $Aut(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^\times$.

Example. The automorphism group of $\mathbb{Z}$ is $\{\pm 1\}$. We can see this since any automorphism must send 1 to ± 1.

There is a natural map $\Theta\colon G \to Aut(G)$. This assigns to a group element a, the function θ_a. This is the function which conjugates by a, i.e. $\theta_a(g) = aga^{-1}$.

This function is a group homomorphism. It's image is denoted $Inn(G)$. It is easy to check that the kernel of this homomorphism is exactly $Z(G)$.

Example. The automorphism group of S_3 is S_3. Every automorphism is of the form θ_a for some $a \in S_3$.

This is in fact true for all the symmetric groups except S_6.

One can use symmetric groups to better understand quotients. We have the following theorem.

Theorem 3.0.2. *Let G be a group and $\varphi \in Aut(G)$. Let N be a normal subgroup of G. Let $K = \{\varphi(g) \mid g \in H\}$, then $G/N \cong G/K$*

Proof. Define a function $\alpha\colon G \to G/K$ by $\alpha(g) = \varphi(g)K$. It is easy to see that this is a homomorphism and is surjective. The kernel of this homomorphism can be calculated as follows:

$$\begin{aligned}\ker(\alpha) &= \{g \in G \mid \alpha(g) = K\} \\ &= \{g \in G \mid \varphi(g)K = K\} \\ &= \{g \in G \mid \varphi(g) \in K\} \\ &= N.\end{aligned}$$

We can now appeal to the first isomorphism theorem to complete the proof. □

This will help immensely in the next section.

3.1 Finitely generated Abelian groups

Throughout this course we have attempted to classify groups. Earlier we saw that we are able to classify all groups of order < 8. The classification theorem for finite groups is a deep problem. We begin with an easier problem, namely classifying the finitely generated abelian groups. The arguments that follow are essentially based in linear algebra and row reduction for matrices.

Definition 3.1.1. A group G is *finitely generated* if there is a finite set $S \subset G$ such that $G = \langle S \rangle$.

Example. Any cyclic group is finitely generated.

Example. Any finite group is finitely generated. The set G satisfies the property in the definition.

Example. The group $\mathbb{Z}\times\mathbb{Z}$ is finitely generated where the set S is $\{(1,0),(0,1)\}$.

Non Example. The group $\mathbb{R}$ is not generated since any finitely generated group is countable.

Non Example. $\mathbb{Q}$ is not finitely generated.

Proof. Exercise □

We will be proving the following classification result.

Theorem 3.1.2. *Let G be a finitely generated Abelian group. Then $G \cong \mathbb{Z}^r \times \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_l\mathbb{Z}$ where $d_i > 1$ and d_i divides d_{i+1}.*

Assume that G is a finitely generated abelian group with generating set $\{g_1, \ldots, g_n\}$. Then there is a homomorphism $\varphi\colon \mathbb{Z}^n \to G$ given by $(i_1, \ldots, i_n) \mapsto g_1^{i_1} \ldots g_n^{i_n}$. This homomorphism is surjective.

Let K be the kernel of this homomorphism. By the first isomorphism theorem G is isomorphic to $\mathbb{Z}^n/K$ thus if we can understand the subgroups of $\mathbb{Z}^n$ we can understand the finitely generated abelian groups.

Lemma 3.1.3. *Let G_1, G_2 be groups. Let $H_1 \lhd G_1$ and $H_2 \lhd G_2$. Then $H_1 \times H_2 \lhd G_1 \times G_2$ and $(G_1 \times G_2)/(H_1 \times H_2) \cong G_1/H_1 \times G_2/H_2$.*

Proof. We define a homomorphism $\varphi\colon G_1 \times G_2 \to G_1/H_1 \times G_2/H_2$ by $\varphi((g_1, g_2)) = (g_1H_1, g_2H_2)$. The first isomorphism theorem tells us that $\mathrm{Im}(\varphi) \cong G_1 \times G_2/\ker(\varphi)$.

It should be clear that this map is surjective. Thus we must compute the kernel.

$$\begin{aligned}
\ker(\varphi) &= \{(g_1, g_2) \mid \varphi((g_1, g_2)) = (H_1, H_2)\} && \text{By definition of the identity in } G_1/H_1 \times G_2/H_2 \\
&= \{(g_1, g_2) \mid g_1H_1 = H_1, g_2H_2 = H_2\} && \text{By definition of } \varphi. \\
&= \{(g_1, g_2) \mid g_1 \in H_1, g_2 \in H_2\} && \text{Since } g_iH_i = H_i \text{ if and only if } g_i \in H_i. \\
&= H_1 \times H_2.
\end{aligned}$$

□

Let H be the subgroup of $\mathbb{Z}^n$ given by $l_1\mathbb{Z} \times l_2\mathbb{Z} \times \cdots \times l_n\mathbb{Z}$, i.e. the subgroup where the first coordinate is divisible by l_1 and the second coordinate is divisible by l_2 etc. We get the following corollary.

Corollary 3.1.4. $\mathbb{Z}^n/(l_1\mathbb{Z} \times \cdots \times l_n\mathbb{Z}) \cong \mathbb{Z}/l_1\mathbb{Z} \times \cdots \times \mathbb{Z}/l_n\mathbb{Z}$

To prove the above theorem we will show that given a subgroup of $\mathbb{Z}^n$ there is an isomorphism which send it to a subgroup of the form $l_1\mathbb{Z} \times \cdots \times l_n\mathbb{Z}$.

Proposition 3.1.5. *Any subgroup of $\mathbb{Z}^n$ is finitely generated (by at most n generators).*

Proof. We will prove this by induction. For the base case we have already seen that any subgroup of a cyclic group is cyclic.

Let H be a subgroup of $\mathbb{Z}^n$. Define $F = \{a_1 \mid (a_1, a_2, a_3, \ldots, a_n) \in H\}$.

There are two cases depending whether $F = \{0\}$. Suppose that $F \neq \{0\}$. Then there is an element of F which is positive. Let f be the minimal such element, the first entry of every element of H is divisible by f, otherwise, we can use the Euclidean algorithm to find a smaller such f. Let $b = (f, b_2, \ldots, b_n)$. Let $(a_1, \ldots, a_n) \in H$, then $(a_1, \ldots, a_n) = s(f, b_2, \ldots, b_n) + (0, a_2 - sb_2, \ldots, a_n - sb_n)$.

Let N be the subset of H where the first entry is 0. I.e. $N = \{(a_1, a_2, \ldots, a_n) \in H \mid a_1 = 0\}$. We can see that this is a subgroup of H and that the set $\{(a_2, \ldots, a_n) \mid (0, a_2, \ldots, a_n) \in H\}$ is a subgroup of $\mathbb{Z}^{n-1}$. By induction this is generated by, at most, $n - 1$ elements. We can then take these generators and add b to get a generating set for H. Thus H is generated by, at most, n elements. □

Theorem 3.1.6. *Let H be a subgroup of $\mathbb{Z}^n$. There is an isomorphism which takes H to a subgroup of the form $l_1\mathbb{Z} \times \cdots \times l_n\mathbb{Z}$. Where l_i divides l_{i+1}.*

Note that this accounts for the case the latter l_i are 0.

Proof. Let T be the generating set for $\mathbb{Z}^n$ given by $T = \{e_i\}$ where e_i is an element with zeroes in each position and a 1 in the i-th position.

Let S be a generating set for H. If S has less than n elements, then we will add extra generators of the form $(0, 0, \ldots, 0)$ so that S has size n.

Let A be the $n \times n$ matrix in which the i-th row is the i-th elements of S. I.e. If $S = \{(a_{11}, \ldots, a_{1n}), (a_{21}, \ldots, a_{2n}), \ldots, (a_{n1}, \ldots, a_{nn})\}$. Then

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

We can now apply the following operations

1. Add row i to row j for $i \neq j$.
2. Multiply row i by -1.
3. Switch row i and row j.
4. Add column i to column j.
5. Multiply column i by -1.
6. Switch column i and column j.

We will show that with these operations we can change the matrix A into a diagonal matrix where each entry divide the subsequent entry.

The above operations have the following affects on $T = \{t_1, \ldots, t_n\}$ and $S = \{s_1, \ldots, s_n\}$.

1. Replace s_j by $s_j + s_i$.
2. Replaces s_i by $-s_i$.
3. Switch s_i and s_j.
4. Replace t_j by $t_j + t_i$.
5. Replaces t_i by $-t_i$.
6. Switch t_i and t_j.

Thus by choosing a new generating set for H and applying an isomorphism φ of $\mathbb{Z}^n$ we will have transformed to a subgroup where each element of the generating set is given by an element with at most one non-zero entry. This shows that $\varphi(H) = l_1\mathbb{Z} \times \cdots \times l_n\mathbb{Z}$.

We now complete the proof that we can reduce to a diagonal matrix. By induction, it is enough to show that we can reduce A to a matrix of the form $\begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}$ where B is an $n-1 \times n-1$ matrix and a divides every entry of B.

If the matrix is all zeroes, then we are done. Otherwise, switch rows and columns so that the top left entry is non-zero and has smallest absolute value. By multiplying the first row by -1 we can assume the top left entry is positive.

We can now add rows and columns to reduce all elements of the first row and the first column to be non-negative and smaller than the top left entry. If any element is non-zero switch this to the top left entry and repeat. This will result in the first row and column being zero except for the first entry. With this process we can assume that the top left entry is the smallest non-zero entry of the matrix.

We have now reduced to the matrix $\begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}$ we must show that a divides every entry of B. Suppose that this is not the case, we can assume by switching rows and columns that it is the top left entry of B.

In this case add row 1 to row 2. By taking column 1 away from column 2 we can reduce the (2, 2) element to less than a. We then switch rows and columns to place this in the top left entry and repeat the above procedure. This process will only be done a finite number of times since A is an integer matrix. □

Example. Let H be the subgroup of $\mathbb{Z}^3$ generated by $\{(15,3,6),(12,24,12),(18,18,36)\}$. Then A is the matrix given below we apply the operations to get a diagonal matrix.

$$A = \begin{bmatrix} 15 & 3 & 6 \\ 12 & 24 & 12 \\ 18 & 18 & 36 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 15 & 6 \\ 24 & 12 & 12 \\ 18 & 18 & 36 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 15 & 6 \\ 0 & -108 & -36 \\ 0 & -102 & -12 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & -108 & -36 \\ 0 & -102 & -12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & -108 & -36 \\ 0 & -102 & -12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 12 & 102 \\ 0 & -36 & -108 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 12 & 102 \\ 0 & 0 & 198 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 12 & 6 \\ 0 & 0 & 198 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 12 \\ 0 & 198 & 0 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 198 & -396 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -396 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 396 \end{bmatrix}$$

Thus, we see that $\mathbb{Z}^3/H$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/396\mathbb{Z}$.

4 Group actions

At this point we have met many examples of groups and hopefully many of them have been interesting to you. Some of these groups can be seen as invertible maps from a space to itself.

Example. The diherdral group D_n is the group of symmetries of the regular n-gon.

Example. The group $GL_n(\mathbb{R})$ is the group of invertible linear transformations from $\mathbb{R}^n \to \mathbb{R}^n$.

Example. The symmetric group is the group of maps from a set of size n to itself.

Every can be realised as a group of symmetries of some object. This idea leads us to the group actions.

Definition 4.0.1. A *left action* of a group G on a set S is a map

$$\rho\colon G \times S \to S,$$

satisfying the following conditions,

- $\rho(e, s) = s$ for all $s \in S$.
- $\rho(g, \rho(h, s)) = \rho(gh, s)$ for all $g, h \in G$ and $s \in S$.

We have mentioned a few examples above however here are a few more.

Example. $\mathbb{Z}$ acts on $\mathbb{R}$ by $\rho(n, r) = n + r$.

Example. Let V be a vector space and $v \in V$ be a vector. Then $\mathbb{Z}$ acts on V by $\rho(n, w) = w + nv$.

Example. The group S_n acts on a set with n elements by $\rho(\sigma, k) = \sigma(k)$.

Example. Any group G acts on itself by left multiplication $\rho(g, h) = gh$.

Example. Any group G acts on itself by conjugation $\rho(g, h) = ghg^{-1}$.

Example. Any group G acts on $\mathcal{P}(G)$ by left multiplication $\rho(g, X) = \{gx \mid x \in X\}$.

Example. Any group G acts on $\mathcal{P}(G)$ by conjugation $\rho(g, X) = \{gxg^{-1} \mid x \in X\}$.

Example. Let G be a group and H be a subgroup. Then G acts by left multiplication on G/H via $\rho(g, g'H) = gg'H$.

Example. Let G be a group and X be a set upon which G acts. Then there is an action of G on $\mathcal{P}(X)$ via $\rho(g, Y) = \{\rho(g, y) \mid y \in Y\}$.

A profitable way of understanding groups is via their actions.

Given a group G and an action of G on a set S. We can associate to each group element g a function

$$\begin{aligned} \rho_g\colon S &\to S \\ s &\mapsto \rho(g, s) \end{aligned}$$

Since ρ is an action of G on S we see that $\rho_g \circ \rho_h = \rho_{gh}$. This allows us to see that ρ_g is a bijection for all g since it has a two-sided inverse, namely $\rho_{g^{-1}}$. Thus, we get a map $G \to Sym(S)$.

Proposition 4.0.2. *Let G be a group acting on a set S. Then the function*

$$\begin{aligned} G &\to Sym(S) \\ g &\mapsto \rho_g \end{aligned}$$

is a homomorphism.

Proof. We must check that $\rho_{gh} = \rho_g \circ \rho_h$. Let us examine $\rho_g(\rho_h(s))$ foe $s \in S$. We get the following

$$\begin{aligned} \rho_g(\rho_h(s)) &= \rho_g(\rho(h,s)) && \text{By definition of } \rho_h(s) \\ &= \rho(g, \rho(h,s)) && \text{By definition of } \rho_g. \\ &= \rho(gh, s) && \text{By definition of a group action.} \\ &= \rho_{gh}(s) && \text{By definition of } \rho_{gh}. \end{aligned}$$

□

This shows us that defining an action of G on S is equivalent to giving a homomorphism $G \to Sym(S)$.

4.1 Orbits and stabilisers

Throughout, let G be a group acting on a set S.

Definition 4.1.1. The *orbit* of $s \in S$ is the set

$$\mathcal{O}(s) = \{t \in S \mid \exists g \in G \text{ such that } t = \rho(g,s)\}.$$

The *stabiliser* of $s \in S$ is the set

$$Stab(s) = \{g \in G \mid \rho(g,s) = s\}.$$

The *fix point set* of $g \in G$ is the set

$$Fix(g) = \{s \in S \mid \rho(g,s) = s\}.$$

Example. When S_n acts on $\{1, \ldots, n\}$ there is one orbit. We also have

$$Stab(i) = Sym(\{1, \ldots, n\} \setminus \{i\}) cong S_{n-1}.$$

Example. Let S_n act on the subsets of $\{1, \ldots, n\}$. There are $n+1$ orbits, each corresponding to $|Y|$ for $Y \subset X$. Let Y be a set of size k. Then

$$Stab(Y) \cong S_k \times S_{n-k}.$$

Example. Let $SL_2(\mathbb{C})$ act on $M_{22}(\mathbb{C})$ via $\rho(A, M) = AMA^{-1}$.

Each orbit has a representative of the form

$$M = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \qquad \lambda, \mu \in \mathbb{C}$$

if the matrix is diagonalizeable or it has a representative of the form

$$M = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \qquad \lambda \in \mathbb{C}$$

if the matrix is not diagonalizeable.

Example. Let S be the set of colouring of the edges of a pentagon red or blue. There is an action of D_5 on S. As there are 5 edges $|S| = 2^5 = 32$. There are 8 orbits represented by

$$BBBBB \quad BBBBR \quad BBBRR \quad BBRBR$$

$$BBRRR \quad BRBRR \quad BRRRR \quad RRRRR$$

Example. When a group G acts on itself by left multiplication i.e. $\rho(g, h) = gh$. There is one orbit. The stabiliser of any element is $\{e\}$.

Example. Let G act on itself by conjugation. Then $Stab(h) = \{g \in G \mid ghg^{-1} = h\}$, this is known as the *centralizer of* h and is denoted $C_G(h)$. The orbit of h is the conjugacy class of h denoted $\mathcal{C}_h$.

We call an action with one orbit *transitive*. We call an action where the stabiliser of any element is $\{e\}$ *free*.

Let G be a group acting on a set A.

Proposition 4.1.2. *The orbits of an action partition the set.*

Proof. Exercise □

Proposition 4.1.3. *The stabilisers of an action are subgroups.*

Proof. Firstly, the identity is in $Stab(a)$ since $\rho(e, a) = a$ for all $a \in A$.

Given $g, h \in Stab(a)$ we have the following.

$$\rho(gh, a) = \rho(g, \rho(h, a)) = \rho(g, a) = a.$$

Thus, $gh \in Stab(a)$.

Finally if $g \in Stab(a)$, then $a = \rho(g, a)$. We get the following

$$\rho(g^{-1}, a) = \rho(g^{-1}, \rho(g, a)) = \rho(g^{-1}g, a) = \rho(e, a) = a.$$

Thus, $g^{-1} \in Stab(a)$. □

We are now ready to state one of the key theorems about group actions.

Theorem 4.1.4. *The Orbit Stabiliser Theorem*

Let G be a group acting on a set S and let $s \in S$. Then there is a bijection

$$\Phi \colon G/Stab(S) \to \mathcal{O}(s).$$

Given by $\Phi(gStab(s)) = \rho(g, s)$.

Proof. There are three steps. First we must show that Φ is well defined. I.e. if $gStab(s) = hStab(s)$, then $\rho(g, s) = \rho(h, s)$. □

Corollary 4.1.5. *Let G be a finite group, then $|\mathcal{C}_h| = |G : C_G(h)|$.*

We can now also immediately see that the size of an orbit divides the size of G.

Example. Determine the number of conjugates of (123) in S_5

Proposition 4.1.6. *A group of order p^r has non-trivial centre.*

Proof. Exercise □

Proposition 4.1.7. *A group of order p^2 is isomorphic to either $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p^2\mathbb{Z}$.*

Proof. Exercise □

Theorem 4.1.8. *Cauchy's theorem*
Let p be a prime number dividing $|G|$. Then there is an element of G of order p.

Proof. Let $A = \{(g_1, \ldots, g_p) \in G^p \mid g_1 g_2 \ldots g_p = e\}$. □

4.2 Orbit Counting

A good use of group actions is understanding how many different ways there are of achieving a task. For instance in in the following example there are 8 orbits.

Example. Let S be the set of colouring of the edges of a pentagon red or blue. There is an action of D_5 on S. As there are 5 edges $|S| = 2^5 = 32$. There are 8 orbits represented by

$$BBBBB \quad BBBBR \quad BBBRR \quad BBRBR$$

$$BBRRR \quad BRBRR \quad BRRRR \quad RRRRR$$

This is telling us that there are essentially 8 different ways to colour the edges of a pentagon with two colours.

In this example it is possible to write down the 32 possibilities and check their equivalences. But for larger problems this is just not viable. Imagine you were asked a similar question but for a decagon with 6 colours, there are now 6^10 possible colourings and understanding the orbits can be tricky. Forunately we have the following theorem, which is usually incorrectly attributed to Burnside to help with this. We need one lemma first.

Lemma 4.2.1. *Let G be a finite group acting on a set S. Suppose that s, t are in the same orbit. Then $|Stab(s)| = |Stab(t)|$.*

Proof. Let $g \in G$ be such that $\rho(g, s) = t$. Let h be an element of $Stab(s)$. Then we get the following

$$\begin{aligned}
\rho(ghg^{-1}, t) &= \rho(gh, \rho(g^{-1}, t)) \\
&= \rho(gh, s) \\
&= \rho(g, \rho(h, s)) \\
&= \rho(g, s) \qquad \text{since } h \in Stab(s). \\
&= t.
\end{aligned}$$

Thus, $ghg^{-1} \in Stab(t)$. Similarly we see that for $k \in Stab(t)$ we have that $g^{-1}kg \in Stab(s)$. Thus we have a bijection of these sets. □

Theorem 4.2.2. *Let G be a finite group acting on a finite set S. Let N be the number of orbits of G. Then*

$$N = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|.$$

Proof. We will consider the set $A = \{(g, s) \in G \times S \mid \rho(g, s) = s\}$ and count up $|A|$ in two different ways. Then

$$|A| = \sum_{g \in G} |\{s \in S \mid \rho(g, s) = s\}| = \sum_{s \in S} |\{g \in G \mid \rho(g, s) = s\}|.$$

The first and second sums respectively equal

$$\sum_{g \in G} |Fix(g)| \qquad \text{and} \qquad \sum_{s \in S} |Stab(s)|.$$

If the orbits are $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N$ then

$$\sum_{s \in S} |Stab(s)| = \sum_{i=1}^{N} \sum_{s \in \mathcal{O}_i} |Stab(s)| \qquad \text{As orbits partition } S.$$

Using the Orbit-Stabilizer Theorem, this is turn can be rewritten as

$$\sum_{i=1}^{N} \sum_{s \in \mathcal{O}_i} |Stab(s)| = \sum_{i=1}^{N} \sum_{s \in \mathcal{O}_i} \frac{|G|}{|\mathcal{O}_i|} = \sum_{i=1}^{N} |G| = N|G|$$

Hence

$$N|G| = \sum_{g \in G} |Fix(g)|$$

and the result follows. □

4.3 A_n is a simple group

The purpose of studying quotient groups is the hope that by understanding a quotient and a subgroup we can put this information together to understand the group G. One such theorem is the following.

Theorem 4.3.1. *Let G be a group. If $G/Z(G)$ is cyclic, then G is Abelian.*

If one has a group and a normal subgroup, then we can quotient out by the normal subgroup and hopefully have a smaller problem. We use this to break groups into smaller pieces. There are of course groups which have very few normal subgroups. For instance we have seen the following,

Theorem 4.3.2. *Let p be a prime. The only normal subgroups of $\mathbb{Z}/p\mathbb{Z}$ are $\{[0]\}$ and $\mathbb{Z}/p\mathbb{Z}$.*

We call groups with this property simple.

Definition 4.3.3. We say that G is a *simple group* if the only normal subgroups of G are G and the trivial subgroup.

Simple groups are groups that can't be broken apart into smaller pieces. Fortunately, due to a lot of mathematics, simple groups are well understood and classified.

Theorem 4.3.4. *Let G be a finite simple group. Then G is of one of the following 4 types:*

1. *$\mathbb{Z}/p\mathbb{Z}$ for p a prime.*
2. *A_n for $n \geqslant 5$.*
3. *16 families of Lie type.*
4. *26 sporadic groups.*

The proof of this theorem is well beyond the scope of this course.

With simple groups classified one might hope to break groups up into simpler pieces. We do this with composition series.

Definition 4.3.5. A *composition series* for a group G is a sequence of nested subgroups $G = G_0, G_1, \ldots, G_n = \{e\}$ such that $G_{i+1} \lhd G_i$ and G_i/G_{i+1} is simple.

This is the natural notion of breaking a group into smaller pieces. For instance we can see that G_{n-1} is a simple group and thus belongs to the list above. Now if we want to understand G_{n-2} we can look at G_{n-1} and G_{n-2}/G_{n-1}, these are both simple groups and so are on the list. We can put this information together to get a better understanding of G_{n-2} repeating this process we can gain a better understanding of G.

One thing that is useful to know is that such series exist and that they satisfy some uniqueness. This is summarised by the following theorems.

Theorem 4.3.6. *Let G be a finite group. Then there exist a composition series for G.*

Theorem 4.3.7. *Let G be a finite group. Suppose that we have two compositions series $G_0, \ldots, G_n$ and $H_0, \ldots, H_m$ then $n = m$ and the simple groups appearing are the same up to reordering.*

This means that when looking at a group in any decomposition series we need only understand one set of simple groups.

For the remainder of this section we will focus on proving the alternating groups A_n are simple.

We start by proving that A_5 is simple. This proof really is studying the action of A_5 on itself by conjugations. Recall that G acts on itself by conjugation via $\rho(g, h) = ghg^{-1}$. Recall, the orbit τ under this action is called the *conjugacy class* of h and is denoted $\mathcal{C}_h$.

Lemma 4.3.8. *Suppose that N is a normal subgroup of G. Suppose that $h \in H$. Then the conjugacy class of h is contained in N.*

Proof. Since N is normal we see that $gkg^{-1} \in N$ for all $k \in N$ and $g \in G$. In particular, $ghg^{-1} \in N$ for all $g \in G$. Thus the conjugacy class of h is contained in N. □

Corollary 4.3.9. *Let G be a group and N a normal subgroup. Then N is a union of conjugacy classes.*

Proof. By the above lemma, we see that $N = \cup_{h \in N} \mathcal{C}_h$. □

Thus if we wish to understand normal subgroups we can understand conjugacy classes.

Understanding conjugacy classes in A_5 is related to understanding conjugacy classes in S_5. We have already seen that two permutations are conjugate if and only if they have the same cycle type, thus conjugacy in S_5 is easy to understand.

We will now look at the size of conjugacy classes in A_5. To do this we use the orbit stabiliser theorem and to calculate their size, this requires us to understand the size of centralisers in A_5.

To understand the size of centralisers in A_5 we use the following theorem.

Theorem 4.3.10. *Let H be a subgroup of S_n. Then $H \cap A_n = H$ of $|H \cap A_n| = \frac{|H|}{2}$.*

Proof. If $H \subset A_n$, then $H \cap A_n = H$. Suppose that H is not contained in A_n. Let $h \in H$ be an element such that $h \notin A_n$. Then $S_n = A_n \cup hA_n$. We can now see that $H = H \cap (A_n \cup hA_n) = (H \cap A_n) \cup (H \cap hA_n)$. We can also see that $H \cap hA_n = h(H \cap A_n)$. Thus we see that $H \cap A_n$ has index 2 in H and thus has size $\frac{|H|}{2}$. □

Now we have this let us examine the various cycle types of permuations in S_5 and their stabilisers.

σ	Permutations with same cycle type	$\lvert C_{S_5}(\sigma)\rvert$	$C_{S_5}(\sigma)$
e	1	120	S_5
(123)	20	6	$\langle(123),(45)\rangle$
(12)(34)	15	8	$\langle(12),(34),(1324)\rangle$
(12345)	24	5	$\langle(12345)\rangle$

We can now compute $|C_{A_5}(\sigma)|$ in each case above, this allows us to compute the conjugacy class of each permutation in A_5

σ	$\lvert C_{A_5}(\sigma)\rvert$	$\lvert\mathcal{C}_\sigma\rvert$
e	60	1
(123)	3	20
(12)(34)	4	15
(12345)	5	12

We notice that there are 12 5-cycles in the conjugacy class of (12345). This means that there is another conjugacy class of 5-cycles. This is the conjugacy class of (21345). We can see that all the conjugacy classes remain the same except for that of the 5-cycles which split into two conjugacy classes.

σ	$\lvert\mathcal{C}_\sigma\rvert$
e	1
(123)	20
(12)(34)	15
(12345)	12
(21345)	12

If there were a normal subgroup of A_5 then it would be a union of conjugacy classes. Also it would contain e and have order dividing 60. We can see that taking any union of conjugacy classes with the identity cannot produce these properties unless we take all the elements of A_5 or just the identity. This proves that A_5 is a simple group.

5 Rings

5.1 Basic definitions

Rings are much like groups except they have two binary operations. While this may feel like more abstraction you have actually been using rings all your life!

Definition 5.1.1. A *ring* $(R,+,\times)$ is a set R together with two binary operation $+$ and $\times$. These operations satisfy the following axioms:

- $(R,+)$ is an abelian group. Since this is an abelian group, we write 0 or 0_R for the identity element of this group.
- $\times$ is an associative binary operation, i.e. $a\times(b\times c)=(a\times b)\times c$ for all $a,b,c\in R$.
- $\times$ is distributive over $+$, i.e. $a\times(b+c)=(a\times b)+a\times c$, for all $a,b,c\in R$.

Notation 5.1.2. We will often drop $\times$ in notation, ab for $a \times b$. Much like when we dropped the $*$ in group theory.

Remark 5. If the operations $+$ and $\times$ are clear from context we will just write R for $(R, +, \times)$.

Definition 5.1.3. We say that a ring is *commutative* if $\times$ is a commutative binary operation, i.e. $ab = ba$.

Definition 5.1.4. We say that a ring has an *identity*, if there is an element $1_R \in R$ such that $1_R r = r = r1_R$ for all $r \in R$. We also require that $1_R \neq 0_R$.

We will study rings that have an identity also we will mostly be interested in commutative rings although some examples will be non-commutative.

The following simple algebra facts are left as an exercise.

Proposition 5.1.5. *Let R be a ring with an identity and $a, b, c \in R$.*

- *If $a + b = a + c$, then $b = c$.*
- $-(-a) = a$.
- $a \times 0_R = 0_R = 0_R \times a$.
- $-ab = -a(b) = a(-b)$.
- $(-1_R)a = -a = a(-1_R)$.

Example. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are rings with the usual operations of $+$ and $\times$. In each case $0_R = 0$ and $1_R = 1$.

Example. The ring of integers modulo n is a ring with the standard operations of addition and multiplication.

Example. Given two rings $(R, +_R, \times_R)$ and $(S, +_S, \times_S)$. Then the *direct sum*, $R \oplus S$ of R and S is the ring with underlying set $R \times S$ and

$$(r, s) + (r', s') = (r +_R r', s +_S s')$$

and

$$(r, s) \times (r', s') = (r \times_R r', s \times_S s').$$

Example. $C(\mathbb{R})$, the set of continuous functions from $\mathbb{R} \to \mathbb{R}$ is a ring with pointwise addition and pointwise multiplication. $0_{C(\mathbb{R})}$ is the zero function and $1_{C(\mathbb{R})}$ is the constant function 1.

To prove that this is a ring we need the basic calculus fact that if f, g are continuous functions, then $f + g$ and fg are continuous functions.

Example. The set of even integers $2\mathbb{Z}$ is a commutative ring although it does not have an identity.

Example. The set of $n \times n$ matrices with real coefficients is a non commutative ring. It has an identity, namely the identity matrix.

Example. We can extend the above example to $n \times n$ matrices with coefficients in any ring R. This will be denoted $M_n(R)$.

Example. The power set $\mathcal{P}(X)$ of a set X with the operations symmetric difference $\triangle$ and intersection $\bigcap$ forms a commutative ring.

Recall $A \triangle B = A \setminus B \cup B \setminus A$.

Definition 5.1.6. Let R be a ring. A subset S of R is a *subring* if

1. $1_R \in S$,
2. for all $r, s, t \in S$ we have that $r - st \in S$.

Equivalently S contains the identity and is a ring with the operations coming from R.

Example. $\mathbb{Z}$ is a subring of $\mathbb{Q}$ which is a subring of $\mathbb{R}$ which is a subring of $\mathbb{C}$.

Example. Matrices with real coefficients form a subring of the matrices with complex coefficients.

There are certain elements in a ring which can be of particular interest. The first is that of a unit.

Definition 5.1.7. A *unit* in a ring R is a non-zero element a such that there exists a b such that $ab = 1 = ba$.

Units are the elements of a ring which have a multiplicative inverse.

Example. In $\mathbb{Z}$ the units are $1, -1$. In $\mathbb{Q}$ the units are all non-zero elements this is also true in $\mathbb{R}$ and $\mathbb{C}$.

Example. For the ring $n \times n$ matrices with real coefficients, the units are the invertible matrices. We met these earlier in the course as $GL_n(\mathbb{R})$.

Example. In the ring $C(\mathbb{R})$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, the units are the functions which are never 0.

Nicely we have the following.

Theorem 5.1.8. *The units in a ring R form a group under multiplication. This group is denoted $R^\times$.*

Proof. Firstly we check that the product of two units is a unit. Let a, a' be units. Thus there are b, b' such that $ba = ab = 1_R = a'b' = b'a'$. We can then see that $aa'b'b = a1_Rb = ab = 1_R$ also $b'baa' = b'1_Ra' = b'a' = 1_R$. Thus aa' is a unit.

The element 1_R is a unit and is the identity for this group.

Multiplication is associative since multiplication in a ring is associative.

Finally the inverse of a unit a is the element b such that $ab = 1_R$. □

We have seen some examples where most elements are units i.e. $\mathbb{Q}$ and $\mathbb{R}$. These rings are known as fields.

Definition 5.1.9. A commutative ring R is a *field* if every non-zero element is a unit.

Example. The rings $\mathbb{Q}$ and $\mathbb{R}$ are fields.

Example. The ring $\mathbb{Z}_7$ is a field. In fact, $\mathbb{Z}_n$ is a field if and only if n is prime.

In the other direction we have zero-divisors.

Definition 5.1.10. A zero-divisor in a ring R is a non-zero element a such that there is are elements b, c such that $ab = 0 = ca$.

Example. The zero divisors in $\mathbb{Z}_6$ are $2, 3, 4$.

Example. The zero-divisors in $C(\mathbb{R})$ are the functions which are zero on (a, b) for some $a, b \in \mathbb{R}$.

Example. In $\mathcal{P}(X)$ every non-empty set is a zero-divisor.

Definition 5.1.11. A ring which contains no zero-divisors is an *integral domain.*

Example. The rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are integral domains.

Example. If R is an integral domain and S is a subring of R, then S is an integral domain.

Example. Any field is an integral domain.

This is a consequence of the following.

Proposition 5.1.12. *An element cannot be both a zero-divisor and a unit. There are rings with elements that are neither a zero-divisor nor a unit.*

Proof. Suppose that a is a unit and also a zero divisor. Thus there are $b, c \neq 0_R$ such that $ab = 1_R$ and $ca = 0_R$. We thus get the following equality $cab = 0_R b = 0_R$ and also $cab = c1_r = c$ and thus $c = 0_R$ giving a contradiction. □

5.2 Polynomial Rings

Given a ring R we have a well defined notion of addition and multiplication we can use this to define new rings such as the ring of matrices with coefficients in R or polynomial rings.

Example. If R is a ring, then we can consider $R[x]$ the ring of polynomials with coefficients in R. If R is commutative, then $R[x]$ is commutative. If R has an identity, then $R[x]$ has an identity, namely the constant polynomial 1_R. Explicitly given two polynomials $p(x) = \sum_i a_i x^i$ and $q(x) = \sum_i b_i x^i$. Then

$$p(x) + q(x) = \sum_i (a_i + b_i) x^i \qquad p(x)q(x) = \sum_k \sum_{i+j=k} a_j b_i x^k.$$

We can define polynomials rings in several variables by

$$R[x_1, \ldots, x_n] = R[x_1, \ldots, x_{n-1}][x].$$

Definition 5.2.1. We say that the *degree* of a polynomial $p(x) = \sum_i a_i x^i \in R[x]$ is $\deg(p) = \max\{i \mid a_i \neq 0\}$.

Note that, in general, we do not have the equality $\deg(p(x)q(x)) = \deg(p(x)) + \deg(q(x))$.

Example. Let $R = \mathbb{Z}_4[x]$ and $p(x) = q(x) = 2x + 1$. Then

$$p(x)q(x) = 4x^2 + 4x + 1 = 1.$$

But $\deg(p(x)) = \deg(q(x)) = 1$ and $\deg(p(x)q(x)) = 0$.

Also note that factorisation is not unique.

Example. The polynomial $x^2 - 1 \in \mathbb{Z}_8[x]$ has the following factorisations

$$x^2 - 1 = (x + 1)(x - 1) = (x - 3)(x - 5).$$

These properties are mysterious at first glance as both are things that naturally work when we consider polynomials with integer or even real coefficients. In fact we can summarise this failure in terms of the original ring R.

Theorem 5.2.2. *If R is an integral domain, then the equality* $\deg(p(x)q(x)) = \deg(p(x)) + \deg(q(x))$ *holds in* $R[x]$.

Proof. Let the $\deg(p(x)) = n$ and $\deg(q(x)) = m$. Then we can look at the coefficient p_n of x^n in $p(x)$ and the coefficient q_m of x^m in $q(x)$. In the product the coefficient of x^{m+n} is $p_n q_m$ which is non-zero since R is an integral domain. □

Theorem 5.2.3. *Let R be a ring. R is an integral domain if and only if $R[x]$ is an integral domain.*

Proof. If R has a zero divisor a, then the constant polynomial a is a zero divisor in $R[x]$.

The previous theorem shows the other direction. □

The failure of factorisation is a little more complicated and is beyond the scope of this course cf. unique factorisation domains. Later we will look at the case of fields.

5.3 Ideals and homomorphisms

From this point onwards all rings will be assumed to be commutative.

In groups we looked at normal subgroups and quotient groups. We will now look at the analogue of this in rings. The corresponding concept is known as an ideal

Definition 5.3.1. A non-empty subset I of a ring R is an ideal if the following are satisfied.

1. If $i, j \in I$, then $i - j \in I$.

2. If $i \in I$ and $r \in R$, then $ri \in I$.

If I is an ideal of R, then we write $I \lhd R$.

Theorem 5.3.2. *Let R be a ring and $I \subset R$. Then I is an ideal if and only if I is a subgroup of $(R, +)$ and $ri \in I$ for all $r \in R, i \in I$.*

Proof. Exercise. □

Definition 5.3.3. Let R be a ring and $a \in R$. The *principal ideal* generated by a, denoted (a) is the smallest ideal containing a. So

$$(a) = \{ra \mid r \in R\}.$$

An ideal I is said to be *principal* if there is an element $a \in R$ such that $I = (a)$.

Example. The sets $\{0_R\}$ and R are always ideals of R.

Example. The $\{f \in C(\mathbb{R}) \mid f(0) = 0\}$ is an ideal in $C(\mathbb{R})$.

Definition 5.3.4. The *ideal generated by* $\{a_1, \ldots, a_k\}$ is

$$(a_1, \ldots, a_k) = \{r_1a_1 + r_2a_2 + \cdots + r_ka_k\}$$

is the smallest ideal containing $a_1, \ldots, a_k$.

Example. Let $R = \mathbb{Z}[x]$ and $I = (2, x)$. This is an example of an ideal which is non-principal.

Proposition 5.3.5. *The ideal in $\mathbb{Z}$ are $n\mathbb{Z}$ for some integer n.*

Proof. Any ideal of $\mathbb{Z}$ as a ring is an additive subgroup. The subgroups of $\mathbb{Z}$ are $n\mathbb{Z}$ for some $n \in \mathbb{Z}$. We must now check that $kr \in n\mathbb{Z}$ for $k \in n\mathbb{Z}$ and $r \in \mathbb{Z}$. This should be clear since the elements of $n\mathbb{Z}$ are the numbers which are divisible by n and since k is divisible by n, kr is also divisible by n. □

Proposition 5.3.6. *R is a field if and only if the only ideals are $\{0\}$ and R.*

Proof. Suppose that R is a field and I is an ideal. Suppose that $I \neq \{0\}$. Let $a \in I$ be an element such that $a \neq 0$. Since R is a field there is a b such that $ab = 1$. Thus, $1 \in I$ and so $r = 1 \times r \in R$ for all $r \in R$, hence $I = R$.

Suppose that the only ideals are $\{0\}$ and R. Then for each non-zero elements $a \in R$ we see that $(a) = R$ since it contains a. Thus, $1 \in (a)$, thus there is an element b such that $ab = 1$. Thus a has an inverse and R is a field. □

In groups we could take quotients by normal subgroups to create new groups. In rings we can do the same thing with ideals.

Definition 5.3.7. Let R be a ring and $I \lhd R$ be an ideal. The *coset* of $r \in R$ is

$$r + I = \{r + i \mid i \in I\}.$$

We denote the *set of cosets* $\dfrac{R}{I}$.

Proposition 5.3.8. *Let R be a ring and $I \lhd R$ be an ideal. $(\dfrac{R}{I}, \oplus, \otimes)$ forms a ring, where*

$$(r + I) \oplus (s + I) = (r + s) + I$$
$$(r + I) \otimes (s + I) = (rs) + I.$$

Proof. Exercise □

Similarly to our study of groups there is a notion of homomorphism between rings.

Definition 5.3.9. Let R and S be rings. A map $\psi \colon R \to S$ is a *ring homomorphism* if the following hold,

1. $\psi(1_R) = 1_S$,
2. $\psi(a +_R b) = \psi(a) +_S \psi(b)$,
3. $\psi(a \times_R b) = \psi(a) \times_S \psi(b)$.

Example. The map $\psi \colon \mathbb{Z} \to \mathbb{R}$ given by $\psi(n) = n$ is a ring homomorphism.

Example. Let R be a commutative ring with an identity and $a \in R$. Then the map $\psi \colon R[x] \to R$ given by $\psi(p) = p(a)$ is a ring homomorphism.

Sometimes we will just say homomorphism when it is clear we are talking about rings.

Proposition 5.3.10. *Let R, S be rings and $\psi \colon R \to S$ be a ring homomorphism. Let $r \in R$ and n be an integer. Then*

1. *$\psi(0_R) = 0_R$*
2. *If R has an identity and S is an integral domain, then either $\psi(a) = 0$ for all $a \in R$ or $\psi(1_R) = 1_S$.*
3. *$\psi(nr) = n\psi(r)$*
4. *If $n > 0$, then $\psi(r^n) = \psi(r)^n$.*

Proof. Exercise □

Definition 5.3.11. A *ring isomorphism* is a bijective ring homomorphism. We say two rings are *isomorphic* if there is an isomorphism between them. If R and S are isomorphic we write $R \cong S$.

Example. Complex conjugation is a ring isomorphism from $\mathbb{C}$ to $\mathbb{C}$.

Once again we have an notion of kernel and image.

Definition 5.3.12. Let R, S be rings and $\psi\colon R \to S$ be a ring homomorphism.
The *image* of ψ is the set $\mathrm{Im}(\psi) = \{s \in S \mid s = \psi(r) \text{ for some } r \in R\}$.
The *kernel* of ψ is the set $\ker(\psi) = \{r \in R \mid \psi(r) = 0_S\}$.

Proposition 5.3.13. *Let R, S be rings and $\psi\colon R \to S$ be a ring homomorphism. Then $\mathrm{Im}(\psi)$ is subring of S and $\ker(\psi)$ is an ideal of R.*

Proof. Exercise □

Example. Let R be a ring and $\psi\colon R[x] \to R$ be the homomorphism given by $\psi(p) = p(a)$. Then $\ker(\psi) = (x - a)$ and $\mathrm{Im}(\psi) = R$.

Example. The kernel and image of the map $\psi\colon \mathbb{Z} \to \mathbb{Z}_n$ given by $x \mapsto \bar{x}$ are

$$\ker(\psi) = n\mathbb{Z} \text{ and } \mathrm{Im}(\psi) = \mathbb{Z}_n.$$

Example. Consider the homomorphism $\psi\colon \mathbb{R}[x] \to \mathbb{C}$ given by $\psi(p) = p(i)$. The kernel $\ker(\psi) = (x^2 + 1)$ and $\mathrm{Im}(\psi) = \mathbb{C}$.

In groups we had isomorphism theorems which allowed us to understand the image of a homomorphism through quotients. We have an almost exact replica of this theorem for rings.

Theorem 5.3.14. *Let R, S be rings and let $\psi\colon R \to S$ be a ring homomorphism. Then*

1. $\ker(\psi) \lhd R$,
2. $\mathrm{Im}(\psi)$ *is a subring of* S,
3. $\dfrac{R}{\ker(\psi)} \cong \mathrm{Im}(\psi)$.

Proof. content... □

Example. Let R be a ring and $\psi\colon R[x] \to R$ be the homomorphism given by $\psi(p) = p(a)$. Then the isomorphism theorem tells us that $\dfrac{R[x]}{(x-a)} \cong R$.

Example. The isomorphism theorem applied to $\psi\colon \mathbb{Z} \to \mathbb{Z}_n$ given by $\psi(x) = \bar{x}$ says that $\dfrac{\mathbb{Z}}{n\mathbb{Z}} \cong \mathbb{Z}_n$.

Example. The isomorphism theorem applied to the map $\psi\colon \mathbb{R}[x] \to \mathbb{C}$ given by $\psi(p) = p(i)$ tells us that $\dfrac{\mathbb{R}[x]}{(x^2+1)} \cong \mathbb{C}$.

Given 2 ideals there are several operations we can perform to produce a new ideal.

Definition 5.3.15. Let R be a ring and I and J be ideals. Then the following are also ideals.

1. The *sum* of I and J is
$$I + J = \{i + j \mid i \in I, j \in J\}.$$
2. The *intersection* of I and J is
$$\{r \mid r \in I \text{ and } r \in J\}.$$
3. The *product* of I and J is
$$IJ = \{\sum_k i_k j_k \mid i_k \in I, j_k \in J, k \in \mathbb{N}\}.$$

We say that I and J are *coprime* if $I + J = R$.

Example. If m, n are integers and $I = (m)$ and $J = (n)$, then

1. the sum $I + J = (hcf(m, n))$.
2. the intersection $I \cap J = (lcm(m, n))$
3. the product $IJ = (mn)$.

Thus we can see that the ideal (m) and (n) are coprime if and only if there highest common factor is 1, i.e. m and n are coprime in the usual sense.

Proposition 5.3.16. *Let I and J be ideals. Then $IJ \subset I \cap J$. Moreover if I and J are coprime, then $IJ = I \cap J$.*

Proof. content... □

We are now ready to discuss the Chinese remainder theorem.

Theorem 5.3.17. *Let I and J be coprime ideals of a ring R. Then the map*
$$\psi\colon \frac{R}{I \cap J} \to \frac{R}{I} \oplus \frac{R}{J} \qquad \text{given by} \qquad r + I \cap J \mapsto (r + I, r + J)$$
is an isomorphism

Proof. content... □

Noting that with the above proposition we get the following corollary.

Corollary 5.3.18. *Let I and J be coprime ideals of a ring R. Then*
$$\frac{R}{IJ} \cong \frac{R}{I} \oplus \frac{R}{J}.$$

This is particularly useful in the integers.

Corollary 5.3.19. *If m and n are coprime, then*

$$\mathbb{Z}_{mn} \cong \mathbb{Z}_n \oplus \mathbb{Z}_m.$$

Given a list of congruences we can find the congruence in a product.

Example. Find $x \mod 165$ given that

$$\begin{aligned} x &\equiv 1 \mod 11 \\ x &\equiv 2 \mod 15 \end{aligned}$$

5.4 Prime and maximal ideals

Definition 5.4.1. Let R be a ring. An ideal I is proper is $I \subsetneq R$.

Note that I is a proper ideal if and only if the quotient R/I is non-trivial.

Definition 5.4.2. Let R be a commutative ring. We say that an ideal is prime if it is a proper ideal and given $a, b \in R$ such that $ab \in I$, then $a \in I$ or $b \in I$.

Prime ideals correspond to nice properties of the quotient R/I. Namely,

Theorem 5.4.3. *Let I be a prime ideal in the ring R. Then R/I is an integral domain.*

Proof. Suppose that $ab + I = I$. This is equivalent to saying that $ab \in I$. Since I is prime we see that $a \in I$ or $b \in I$. This is the same as saying that $a + I = I$ or $b + I = I$, i.e. at least one of them is zero in R/I. □

Definition 5.4.4. A proper ideal I is *maximal* if for all ideals J with $I \subset J$ either $I = J$ or $J = R$.

Theorem 5.4.5. *If I is a maximal ideal of R, then R/I is a field.*