Practice Midterm

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1. Let G be a group. For each of the following define the term in italics
   (a) The *subgroup generated by a set* S.
   (b) An *equivalence relation* on a set X.
   (c) ϕ is a *homomorphism*.
   (d) The *order* of g.
   (e) The *group of units* of $\mathbb{Z}/n\mathbb{Z}$.

2. Let G and H be finite groups. Let $\phi\colon G \to H$ be a homomorphism. Define $I_G = \{g \in G \mid g^2 = e\}$ and $I_H = \{h \in H \mid h^2 = e\}$.
   (a) Show that $\phi(I_G) \subset I_H$.
   (b) Show that if ϕ is an isomorphism, then $|I_G| = |I_H|$.
   (c) Let D_4 be the group of symmetries of the square. Let r be a rotation and s be a reflection. Fill in the Cayley table for D_4 below.
   (d) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quarternions. Fill out the Cayley table for Q_8 given below.
   (e) By considering the sets I_G for Q_8 and D_4. Show that these groups are not isomorphic.

3. (a) Let $\phi\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be a homomorphism. Show that ϕ is completely determined by $\phi([1])$.
   (b) Show that any homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is of the form $[a] \mapsto [ka]$.
   (c) Let $k \in \mathbb{Z}$. Show that the function $[a] \mapsto [ka]$ is a homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.
   (d) Show that the order of $[k]$ in $\mathbb{Z}/n\mathbb{Z}$ is $\dfrac{n}{\gcd(k, n)}$.
   (e) Show that the above map is an isomorphism if $\gcd(k, n) = 1$.
   (f) Deduce that $Aut(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{\times}$.

$\cdot$	e	r	r^2	r^3	s	sr	sr^2	sr^3
e	e	r	r^2	r^3	s	sr	sr^2	sr^3
r	r	r^2	r^3	e				
r^2	r^2	r^3	e	r				
r^3	r^3	e	r	r^2				
s	s							
sr	sr							
sr^2	sr^2							
sr^3	sr^3							

$\cdot$	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i							
j	j							
k	k							
-1	-1							
$-i$	$-i$							
$-j$	$-j$							
$-k$	$-k$							