Sheet 2: Modular arithmetic, binary operations and groups

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1. Let $n \geqslant 2$. Let $\sim$ be the relation on $\mathbb{Z}$ given by $l \sim m$ if n divides $l - m$. Show that $\sim$ is an equivalence relation on $\mathbb{Z}$.

2. Let p be a prime. Show the product of two non-zero elements in $\mathbb{Z}/p\mathbb{Z}$ is non-zero.

3. Find a multiplicative inverse for each non-zero element of $\mathbb{Z}/7\mathbb{Z}$.

4. For the following sets S and binary operations $*$, state whether a) it is associative, b) it is commutative, c) it has in identity, d) it has inverses

   (a) $S = \mathbb{N}$ and $n * m = n + m$

   (b) $S = \mathbb{N}$ and $n * m = \max\{n, m\}$

   (c) $S = \mathbb{Z}$ and $n * m = n + m + 1$

   (d) $S = \{f : \mathbb{R} \to \mathbb{R}\}$ and $f * g = f \circ g$

   (e) $S = \mathbb{R}^3$ and $v * w = v \times w$ here $\times$ denotes the cross product of two vectors.

   (f) $S = \mathbb{N}$ and $n * m$ is the lowest common multiple of n and m.

5. An affine transformation of $\mathbb{R}^2$ is a function of the form $v \mapsto Av + b$ where A is an invertible 2×2 matrix and b is a vector in $\mathbb{R}^2$. Show that the composition of two affine transformations is an affine transformation. Show further that the set $AGL_2(\mathbb{R})$ of affine transformations forms a group.

6. Show that $GL_2(\mathbb{R})$ is not abelian.

7. Let G be a group. Show that if $a^2b^2 = (ab)^2$, then $ab = ba$.

8. With standard addition show that the set of even integers form a group while the set of odd integers do not.

9. Let $*$ be a binary operation on $(1, \infty)$ given by

$$a * b = ab - a - b + 2.$$

(a) Show that $*$ is an associative binary operation.

(b) Show that 2 is the identity for $*$

(c) Find the inverse of a.