Sheet 3: Cayley tables, cyclic groups, and dihedral groups

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1. Recall that given two groups $(G, *_G)$ and $(H, *_H)$ we can form the direct product $G \times H$ with the operation $(g, h) * (g', h') = (g *_G g', h *_H h')$. Show that this group is Abelian if and only if G and H are Abelian.

2. Show that the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not cyclic.

3. There are 56 Latin squares on the elements $\{e, a, b, c, d\}$ whose first row and column are depicted below.

$\cdot$	e	a	b	c	d
e	e	a	b	c	d
a	a				
b	b				
c	c				
d	d				

   Complete this table in two ways. Firstly complete it to a Cayley table for a group and show that this is indeed the Cayley table for some group. Secondly complete this to a Latin square which does not represent a group. Explain why the Latin square does not correspond to any group.

4. The following table represents a group G (you do not have to prove this)

$\cdot$	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	e	h	g	f	d	c	b
b	b	c	d	f	g	h	e	a
c	c	b	a	e	h	g	f	d
d	d	f	g	h	e	a	b	c
f	f	d	c	b	a	e	h	g
g	g	h	e	a	b	c	d	f
h	h	g	f	d	c	b	a	e

   (a) Find the inverse of each element of the group.

   (b) Find all the elements which commute with every element of G.

   (c) Find the order of each element of the group G.

(d) Show that by relabelling this group is equal to $\{e, b, b^2, b^3, a, ab, ab^2, ab^3\}$.

(e) Show that $ba = ab^3$.

5. Let g be an element of a finite group. Show that the $o(g^k) \leqslant o(g)$.

6. The Euclidean algorithm shows that given two integers n, k there are integers r, s such that $nr + sk = \gcd(n, k)$. Using this fact or otherwise show that:

   (a) If $o(g), k$ are coprime, then $o(g^k) = o(g)$.

   (b) (Optional)In general, $o(g^k) = \dfrac{o(g)}{\gcd(k, o(g))}$.

7. Let g, h be elements of a group. Show that $o(g) = o(h^{-1}gh)$.

8. By considering the matrices given in class or otherwise. Find two elements g, h of a group such that g and h have finite order by the product gh has infinite order.

9. Let D_5 be the group of symmetries of a pentagon. There are 10 elements of D_5.

   (a) Write down the Cayley table for D_5.

   (b) Find two elements a, b of D_5 such that:

      - Both a and b have order 2.
      - Every element of D_5 can be written as a product of powers of a, b. That is, for each element in your Cayley table find a way of writing it as $a^{n_1}b^{n_2}\ldots b^{n_k}$. The set $\{a, b\}$ is called a *generating set* for D_5.

   (Optional) Why can we take all the n_i to be less than 2?