Sheet 4: Symmetric groups, permutations, cycles types and transpositions

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1. Let α, β, γ be permutation on the set $\{1, \ldots, 10\}$ given as follows.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 10 & 6 & 4 & 1 & 2 & 5 & 9 & 8 & 7 \end{pmatrix}$$
$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$
$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 3 & 4 & 5 & 2 & 6 & 1 & 8 & 9 & 7 \end{pmatrix}$$

   (a) Find the order of α, β, γ.
   (b) Write $\alpha^{-1}, \alpha\beta\gamma, \gamma\beta\gamma^{-1}$ as products of disjoint cycles.
   (c) Write α as a product of transpositions.

2. Let α be a permutation on the set $\{1, \ldots, 9\}$ given by

$$\alpha = (1\,3\,5\,7)(1\,2\,6)(3\,7\,9)(5\,4\,3)$$

   Write α as a product of disjoint cycles.

3. Suppose that S is a set of size n. Suppose σ is a permutation of order 11. Show that $n \geqslant 11$.

4. Let $\sigma \in S_n$ and $(1\,2\,\ldots\,k) \in S_n$ be a k-cycle. Show that

$$\sigma(1\,2\,\ldots\,k)\sigma^{-1} = (\sigma(1)\,\sigma(2)\,\ldots\,\sigma(k)).$$

   Where $(\sigma(1)\,\sigma(2)\,\ldots\,\sigma(k))$ is another k-cycle.

5. Let $\alpha, \beta, \gamma \in S_5$ be as follows

$$\alpha = (1\,2)(3\,4)$$
$$\beta = (1\,3\,5)(2\,4)$$
$$\gamma = (1\,4\,2\,5)$$

   Find $(\alpha^5\gamma^3\beta^5\alpha^3\beta^3\alpha\beta\gamma\alpha)^2$. (Hint: Use the previous question and the fact that $\alpha = \alpha^{-1}, \beta^5 = \beta^{-1}$ and $\gamma^3 = \gamma^{-1}$.)

6. Write $(1\,2\,3\,4)$ as a product of transpositions.
7. (Optional) Find permutations $\alpha, \beta \in S_5$ such that they both have order 2 and the product $\alpha\beta$ has order 5.

   Can you find similar permutations in S_n?