Sheet 5: Parity of permutations, the alternating group and subgroups

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1. Write down the possible cycle types of elements of S_5. Which cycle types are in A_5? Write down the number of each cycle type in S_5. Check that you get 120 elements in total, 60 of which are in A_5.
2. Let R be the relation on S_n given by $\sigma R \tau$ if σ and τ are conjugate. Prove that R is an equivalence relation.
3. Show that a cycle of even length is an odd permutation.
4. Show that a cycle of odd length is an even permutation.
5. Show that a permutation with odd order must be an even permutation.
6. Let $n > 3$ and $\sigma \in A_n$. Is it true that σ has odd order? Give a proof or find a counterexample.
7. Let A and B be groups. Prove that for each of the following H is a subgroup of the direct product $A \times B$.
   (a) $H = \{(a, e) \mid a \in A\}$
   (b) $H = \{(e, b) \mid b \in B\}$
   (c) Assuming that $A = B$. $H = \{(a, a) \mid a \in A\}$.
8. For each of the following find an example of a group G with a subgroup H satisfying the following properties. Give a proof that they satisfy the property in question. You may use the results from the previous question.
   (a) G is infinite and H is infinite and cyclic.
   (b) G is infinite and H is finite and cyclic.
   (c) G is non-abelian and H is abelian.
   (d) G is Abelian and H is not cyclic.
9. Consider the Cayley table from problem sheet 3. This group has 10 subgroups of orders $1, 2, 2, 2, 2, 2, 4, 4, 4, 8$. Write down the 10 distinct subgroups.

10. Let G be a group and H, K be subgroups.

   (a) Show that $H \cap K$ is a subgroup.
   (b) Find an example where $H \cup K$ is not a subgroup.
   (c) Let $HK = \{hk \mid h \in H, k \in K\}$. Find an example where HK is not a subgroup.
   (d) Show that if G is Abelian, then HK is a subgroup of G.