Sheet 6: Lagrange's theorem, homomorphisms, isomorphisms, kernels and images

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Definition 0.1. Let G be a group and H a subgroup of G. The *right coset of H corresponding to g* is $Hg = \{hg \mid h \in H\}$.

1. Let $G = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Let $H = \{([0],[0]),([0],[2])\}$ and $K = \{([0],[0]),([3],[0]),([0],[2]),([3],[2])\}$. These are subgroups of G.

   List the left cosets of H. List the left cosets of K.

2. Let A and B be groups. Let $G = A \times B$. Let $H = \{(a,e) \mid a \in A\}$, this is a subgroup of G.

   Show that the left cosets of H in G are in one-to-one correspondence with B.

3. Let $G = S_3$, $H = \{e,(1\,2)\}$ and $K = \{e,(1\,2\,3),(1\,3\,2)\}$.

   List the left and right cosets for H and the left and right cosets for K.

4. Let G be a group and H be a subgroup. Show that two right cosets Hg and Hk are equal if and only if $gk^{-1} \in H$.

5. (a) Use Fermat's little theorem to compute $3^{20} \mod 7$ and $2^{53} \mod 11$.

   (b) Use Euler's theorem to compute $2^{49} \mod 15$ and $4^{38} \mod 21$.

6. Let p be a prime number.

   (a) Let G be a finite group such that $|G| = p$. Let $g, h \in G$. What are the possible orders of $\langle g\rangle \cap \langle h\rangle$?

   (b) Let G be a finite group. Show that the number of elements of order p is divisible by $p-1$.

   (c) Let G be a group of order 35. Show that G has an element of order 5 and an element of order 7.

   (Hint: What are the possible orders for elements of G? What happens if G has an element of order 35?)

7. Let H, K be subgroups of G. Assume that $\gcd(|H|,|K|) = 1$.

   Show that $H \cap K = \{e\}$.

8. Let G be a group such that $|G| = 2n$. Let R be the equivalence relation defined by xRy if $x = y$ or $x = y^{-1}$.

   Show that there are at least two equivalence classes of the form $\{x\}$.

   Deduce that G has an element of order 2.