Sheet 8: Homomorphisms, Isomorphisms, Kernels, Images, Normal subgroups and Quotients

Robert Kropholler

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1. Show that each of the following are homomorphisms. In each case find the kernel and image.
   - $\mathbb{R} \to \mathbb{R} \setminus \{0\}$ given by $x \mapsto 2^x$.
   - $\mathbb{C} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ given by $z \mapsto |z|$. Recall: $|a + bi| = \sqrt{a^2 + b^2}$.
   - $S_3 \to S_4$ given by $\sigma \mapsto \sigma$.
   - $S_3 \to S_4$ given by $\sigma \mapsto (1\,4)\sigma(1\,4)$.
   - $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ given by $([x], [y]) \mapsto ([x + 2y], [y])$.
2. Show that $(\mathbb{Z}/7\mathbb{Z})^\times$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$.
3. Let $Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}$. Recall that this is a normal subgroup of G.

   (a) Suppose that $G/Z(G)$ is cyclic and generated by $hZ(G)$. Show that each element of G is of the form $h^k z$ where $z \in Z(G)$ and $k \in \mathbb{Z}$.

   (b) Show that G is Abelian.
4. Let H be a subgroup of G.

   (a) Show that G is Abelian if and only if $ghg^{-1}h^{-1} = e$ for all $g, h \in G$.

   (b) Suppose that $ghg^{-1}h^{-1} \in H$ for all $g, h \in G$. Show that H is a normal subgroup of G.

   (c) Show that in this case G/H is Abelian.

   (d) Show that if G/H is Abelian, then $ghg^{-1}h^{-1} \in H$ for all $g, h \in G$.
5. Let $D_n = \{r^i s^j \mid i \in \{1, \ldots, n\}, j \in \{0, 1\}\}$ be the dihedral group. Recall that in D_n we have the following equalities $r^n = e = s^2, srs = r^{-1}$. Let k be an integer such that k divides n.

   (a) Show that the function $\phi \colon D_n \to D_k$ given by $\phi(r^i s^j) = r^i s^j$ is a homomorphism.

(b) Show that $H = \langle r^k \rangle$ is a normal subgroup of D_n.

(c) Show that D_n/H is isomorphic to D_k.