Sheet 9: Classification of finitely generated Abelian groups and group actions.

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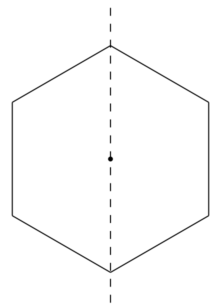
1. (a) Show that if H is a finitely generated subgroup of $\mathbb{Q}$, then H is cyclic. Hint: Show that if H is generated by $\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}$, then $H \leqslant \langle \frac{1}{q_1 \ldots q_n} \rangle$.
   (b) Show that $\mathbb{Q}$ is not cyclic. I.e. show that given any $r \in \mathbb{Q}$ we have that $\langle r \rangle \neq \mathbb{Q}$.
   (c) Deduce that $\mathbb{Q}$ is not finitely generated.
2. How many Abelian groups are there of order 100?
3. Let p be a prime number. Show that any Abelian group of order p^2 is isomorphic to either $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.
4. Let G be a group acting on a set A. Show that the orbits partition A.
5. Let G be a group of order p^2 where p is a prime number. Let G act on G by $\rho(g, h) = ghg^{-1}$. Let $\mathcal{C}_h$ be the orbit of h.
   (a) Show that $\mathcal{C}_e = \{e\}$.
   (b) Show that $|\mathcal{C}_h| = p$ or 1.
   (c) Recall that $Z(G) = \{h \in G \mid gh = hg\}$. Show that $\mathcal{C}_h = \{h\}$ if and only if $h \in Z(G)$.
   (d) Show that $|Z(G)| > 1$.
   (e) Deduce that $G/Z(G)$ is cyclic.
   (f) Deduce that G is either $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.
6. The group D_n acts on the n-gon. Let A be the set of edge colourings of a hexagon by red, green and blue. Let D_6 act on A via the action of D_6 on a hexagon. Let r be the rotation by $\frac{\pi}{3}$ and s be the reflection in the line pictured.



(a) Show that there are 3^6 elements of A.

(b) By considering each of the elements of D_6 calculate how many colourings are fixed by them and complete the following table.

g	$\|Fix(G)\|$
e	3^6
r	
r^2	
r^3	
r^4	
r^5	
s	
sr	
sr^2	
sr^3	
sr^4	
sr^5	

(c) Show that there are 92 possible colourings of a hexagon up to reflections and rotations.