Exam Spring 2017

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This exam is due on Friday 5th at noon. You are allowed to use the lecture notes by Marc Lackenby as well as the supplementary materials written by me (covers of a rose and covers of a wedge sum). You should not use the internet, and should not discuss with fellow students. Please sign your final exam to state that you have adhered to these guidelines.

Throughout you may assume that fundamental group S^1 is $\mathbb{Z}$ and the fundamental group of $S^1 \times S^1$ is $\mathbb{Z}^2$.

1. Let $f \colon S^n \to S^n$ be a continuous function.

   (a) [5 marks] Show that if $f(x) \neq -x$ for all $x \in S^n$ then f is homotopic to the identity map $id(x) = x$.

   (b) [5 marks] Show that if $f(x) \neq x$ for all $x \in S^n$ then f is homotopic to the antipodal map $\alpha(x) = -x$.

   (c) [8 marks] Show that if n is even, then there is an x such that $f(x) = x$ or $f(x) = -x$.

   (You may assume that if n is even, then the identity and antipodal maps are not homotopic.)

   (d) [3 marks] If n is odd find a map such that $f(x) \neq x$ and $f(x) \neq -x$ for all $x \in S^n$.

2. Let $G = \langle a, b \mid a^2 = b^3 \rangle$.

   (a) [2 marks] Define the free group on a set S.

   (b) [3 marks] Show that $Z(F(S)) = \{e\}$ if $|S| > 1$.

   Recall: The centre of a group $Z(H) = \{h \in H \mid gh = hg, \forall g \in H\}$

(c) [2 marks] Given a group H. Give a sufficient and necessary condition for a map $\{a, b\} \to H$ to extend to a homomorphism $G \to H$.

(d) Show that $a^2 \neq e$ in G.
(Hint: Construct a homomorphism $G \to \mathbb{Z}$.)

(e) [3 marks] Show that $Z(G) \neq \{e\}$.
(Hint: $a^2 \in Z(G)$)

(f) [2 marks] Deduce that G is not isomorphic to a free group on a set S, where $|S| \geq 2$.

(g) [4 marks] Show that $ab \neq ba$.
(Hint: Think of a homomorphism $\phi \colon G \to S_3$.)

(h) [2 marks] Deduce that G is not isomorphic to $\mathbb{Z}$.

3. Let M be the Mobius band, $[-1, 1] \times [0, 1]/\sim$, where the equivalence is generated by $(x, 1) \sim (-x, 1)$. Recall, $D^2 = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$.

(a) [3 marks] Compute $\pi_1(M)$.

(b) [2 marks] Compute $\pi_1(D^2)$.
Let $i : S^1 \to D^2$ be the inclusion of the circle as the boundary of the disc. Let $j : S^1 \to M$ be the inclusion of the circle as the boundary of the Mobius band.

(c) [8 marks] Compute the images of i_* and j_*.

(d) [2 marks] State the Seifert van Kampen theorem.

(e) [5 marks] Compute the fudamental group of the quotient spaces

$$D^2 \cup M/(i(x) \sim j(x))$$

and

$$M \sqcup M/(j(x) \sim j(x))$$

These last two spaces are $\mathbb{RP}^2$ and the Klein bottle. You may use either of these facts if you can prove them.

4. (a) [2 marks] State the simplicial approximation theorem.

(b) [7 marks] Show that any 2 maps $S^1 \times S^1 \to S^3$ are homotopic.

(c) [11 marks] Find two maps $(S^1 \times S^1, (1, 1)) \to (S^1 \times S^1 \times S^1, (1, 1, 1))$ which are not homotopic relative to $(1, 1)$.

5. (a) [4 marks] Let X be a space. Define what it means for $\bar{X}$ to be a covering space of X. Define the degree of a covering map.

   Let $X = \mathbb{RP}^2 \vee S^1$ and $Y = S^1 \vee S^1$.

   (b) [10 marks] Draw two different degree 2 covering spaces of Y.

   (c) [6 marks] Draw a degree two covering space of X with fundamental group $\mathbb{Z} * \mathbb{Z}$.

   Draw pictures to describe these covers and then compute the fundamental group.