# Practice Exam

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This exam should be done in 24 hours, you are allowed to use the lecture notes and text books as well as any previous problem sheet answers (reproduce the proofs required.) You should not use the internet, and should not discuss with fellow students.

1. We say that a space $X$ has the fixed point property if for any map $f\colon X \to X$ there is a point $x$ such that $f(x) = x$.
   (a) Show that $D^2$ has the fixed point property.
   (b) Show that if $X$ and $Y$ are homeomorphic and $X$ has the fixed point property, then $Y$ has the fixed point property
   (c) Show that the open disc in $\mathbb{C}$ does not have the fixed point property.

2. (a) Define the free group $F(a,b)$ in terms of words (You do not have to prove that it is a group).
   Let $G = \langle x, y \mid x^2 = y^5 \rangle$.
   (b) Show $F(a,b)$ surjects onto $G$.
   (c) Define the universal property of free groups.
   (d) How many homomorphisms $\psi\colon F(a,b) \to \mathbb{Z}/2\mathbb{Z}$ are there?
   (e) Let $u, v$ be elements of $F(a,b)$ such that $u^2 = v^5$. Show that if $\phi\colon F(a,b) \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism, then $\phi(v) = e$.
   (f) How many homomorphisms $G \to \mathbb{Z}/2\mathbb{Z}$ are there?
   (g) Show there is no surjection $G \to F(a,b)$.

3. Recall $S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subset D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$. Let $X = S^1 \times D^2$ and $Y = S^1 \times S^1$.

   (a) Compute $\pi_1(X, 1)$ and $\pi_1(Y, 1)$.

   Let $i\colon Y \to X$ be the inclusion map i.e. $i(x, y) = (x, y)$.

   (b) Calculate $i_*\colon \pi_1(Y, 1) \to \pi_1(X, 1)$.

   (c) State the Seifert van Kampen Theorem.

   Let $X_1$ and $X_2$ be homeomorphic copies of $X$. Let $i\colon Y \to X_1$ be the map $i(x, y) = (x, y)$. Let $j\colon Y \to X_2$ be the map $j(x, y) = (y, x)$.

   (d) Compute the fundamental group of $X_1 \cup X_2/\sim$, where $i(x, y) \sim j(x, y)$.

   Surprisingly, this space is homeomorphic to $S^3$, you may use this fact if you can prove it.

4. State the simplicial Approximation theorem.

   (a) Show that any two maps $S^1 \times S^2 \to S^4$ are homotopic.

   (b) Find 2 maps $(S^1 \times S^2, (1, 1)) \to (S^1 \times S^1, (1, 1))$ which are not homotopic relative to $(1, 1)$.

5. Let $p : \bar{X} \to X$ be a covering map.

   (a) Define the term regular cover. State a theorem that relates regular covers to normal subgroups.

   Consider the coverings of $S^1 \vee S^1$ shown in Figure 1.

   (b) What are the degrees of these coverings?

   (c) Show that one of these covering corresponds to a normal subgroup.

   (d) Show that in that case the quotient if $\mathbb{Z}/3\mathbb{Z}$

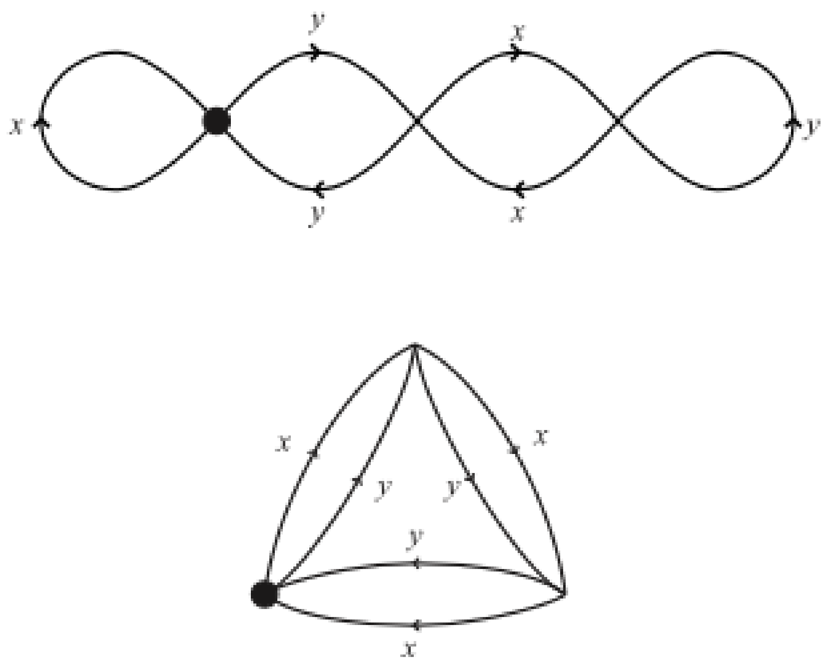


Figure 1: