Constructing Covers of the rose.

Robert Kropholler

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This was discussed in class but does not appear in any of the lecture notes. I will outline the constructions and proof here.

Definition 0.1. Let $F(X)$ be the free group on the set X. Assume $|X| = n$. Let H be any subgroup of $F(X)$, we define the Schrier graph $\mathcal{G}(H)$ as follows:

- The vertices of $\mathcal{G}(H)$ are the left cosets of H in G.
- Two cosets $gH, g'H$ are joined by an edge if there exists an $x \in X$ such that $xgH = g'H$.

It should be noted that the Schrier graph of a normal subgroup N is the Cayley graph of G/N with respect to the generating set X.

Let R_n be a cell complex with 1 vertex and n edges. We identify the fundamental group of R_n with $F(X)$. Pick a bijection between the edges of X and give each edge an orientation.

Proposition 0.2. *The graph $\mathcal{G}(H)$ is a cover of R_n.*

Proof. We can label each edge with an element of X and give it an orientation so that the edge (gH, xgH) points towards xgH. We can then define a map $p : \mathcal{G}(H) \to R_n$. This sends each vertex of $\mathcal{G}(H)$ to the unique vertex of R_n and each edge to the edge with same label by an orientation preserving homeomorphism.

This defines a covering map. We must check that every point has a neighbourhood such that p is a homeomorphism on this neighbourhood.

If we pick a point in the interior of an edge, then we can take an open neighbourhood in the interior of the edge and p will be a homeomorphism on this neighbourhood.

If we take a vertex, then if we take the neighbourhood consisting of a quarter of each edge at this vertex. p will be a homeomorphism restricted to this neighbourhood. □

We can compute the image of $p_*(\mathcal{G}(H), H)$ as follows. We know that $\mathcal{G}(H)$ is a graph and so the fundamental group is free. We also know that it is free on the set of edges not in a maximal tree. Each edge not in the maximal tree has both it's end vertices in the maximal tree. There is a unique embedded path in the tree to the start vertex of such an edge. We follow this path reading the labels of the path and then go along the edge and back in the maximal tree to the base point. This gives us a way to compute a generating set for $p_*(\mathcal{G}(H), H)$.

This set generates H as shown by the following proposition.

Proposition 0.3. *The group $p_*(\mathcal{G}(X), H)$ is exactly the subgroup H of $F(X)$.*

Proof. A loop γ in the graph is labelled by elements of X or X^{-1} given by the orientation on edges. Let us assume the loop γ has label $x_1x_2 \dots x_n$. This corresponds to the $x_1x_2 \dots x_nH = H$, which is equivalent to $x_1x_2 \dots x_n \in H$. This shows that $p_*(\mathcal{G}(H), H) \subset H$. Conversely, given an element of H, written in generators of $F(X)$, we can read this as a path in $\mathcal{G}(H)$ starting at H. This path will give a loop in the graph and thus corresponds to an element of $p_*(\mathcal{G}(H), H)$. This completes the proof that $p_*(\mathcal{G}(H), H) = H$. □

There is one special case where the subgroup H is normal. In this case the Schrier graph is the Cayley graph of $F(X)/H$ with generating set X. This allows for easy computation of covers corresponding to normal subgroups.