Algebraic Topology Problem Sheet 4

Robert Kropholler

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1. Given a group G the centre $Z(G)$ is defined as the subgroup $\{h : gh = hg \text{ for all } g \in G\}$.

   Show that if $|S| > 1$ then $Z(F(S)) = \{e\}$.

2. Let S and T be finite sets. Show $F(S) \cong F(T)$ if and only if $|S| = |T|$. (Hint: show that $\mathrm{Hom}(F(S), G)$ is in bijection with $\mathrm{Map}(S, G)$.) If you are feeling bold, remove the restriction that S and T are finite.

3. Let S and T be disjoint sets. Use universal properties (no presentations) to show that $F(S) * F(T) = F(S \cup T)$.

4. Define the free abelian group on a finite set A to be the group $\mathbb{Z}^{|A|}$. There is a bijection i between A and the set of elements with a 1 in a single position.

   Show that the free abelian group satisfies the following universal property. Given an abelian group G and a map $j\colon A \to G$, there is a unique homomorphism ϕ from the free abelian group on A to G such that $j = \phi \circ i$.

5. (Optional, definitely not easy!)

   Let N be a group with maps $j_1\colon G_1 \to N, j_2\colon G_2 \to N$. Assume N satisfies the universal property of the free product, namely, given maps $f_i\colon G_i \to G$ there is a unique map $\phi\colon N \to G$ such that $\phi \circ j_i = f_i$. Show that N is isomorphic to $G_1 * G_2$.

   A similar result holds for free products with amalgamation.