Practice midterm

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1. Let M be a field containing $\mathbb{Q}$. Let L_1, L_2 be subfields of M.
   (a) Define what it means for $L/\mathbb{Q}$ to be a Galois extension.
   (b) State the fundamental theorem of Galois theory.
   (c) Let L be a Galois extension of $\mathbb{Q}$ and L' be a Galois extension of L. Show, with an example, that $L'/\mathbb{Q}$ need not be Galois.
   (d) Suppose that L_1 and L_2 are finite extensions of $\mathbb{Q}$ and that both are Galois. Let F be the smallest subfield of M containing L_1 and L_2. Show that F is Galois over $\mathbb{Q}$.
   (e) Show that there is a homomorphism $\phi\colon G \to H$, where $G = Aut_K(F)$ and $H = Aut_K(L_1) \times Aut_K(L_2)$. Show that it is injective.

2. (a) Let M be a subfield of $\mathbb{C}$ such that $M/\mathbb{Q}$ is a finite Galois extension. Show that if $[M : \mathbb{Q}]$ is an odd number, then $M \subset \mathbb{R}$.

3. Let L/K be a finite extension.
   (a) When L/K is Galois, define the Galois group, $Aut_K(L)$, of the extension L/K. State clearly the correspondence between subgroups of the Galois group and intermediate field extensions.
   (b) Let M be a field in which the polynomial $x^n - 1 \in \mathbb{Q}[x]$ splits. Let L/M be the splitting field of f over M where $f(x) = x^n - \theta, \theta \in M$. Show that the extension L/M is Galois and that the Galois group $Aut_M(L)$ is cyclic with $|Aut_M(L)|$ dividing n.

4. Let m be a prime. Assume that $F = \mathbb{Q}(\zeta_m)$ i.e. F is the splitting field of $x^m - 1$.

   Assume that $F(\alpha)$ is an extension of F such that $\alpha \notin \mathbb{Q}$ but $\alpha^m \in \mathbb{Q}$.
   (a) Show that if $\alpha^k \in \mathbb{Q}$ for some $k < m$, then $\alpha \in \mathbb{Q}$. Deduce that $\alpha^k \notin \mathbb{Q}$ for all $k < m$.
   (b) Write down the roots of $x^m - \alpha^m$.
   (c) Show that none of the subproducts $(x - \alpha_1) \dots (x - \alpha_k)$ are polynomials in $\mathbb{Q}[x]$ for $k < m$.

(d) Deduce that $x^m - \alpha^m$ is irreducible over F and deduce that $[F(\alpha) : F] = m$.

(e) Show that $Aut_F(F(\alpha))$ is a Galois extensions and the Galois group is $\mathbb{Z}/m\mathbb{Z}$.