Practice Exam

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1. Let $f(x) = x^5 - 1$.
   (a) State the fundamental theorem of Galois theory.
   (b) What are the roots of $f(x)$?
   (c) Show that $f(x) = (x-1)(x^4 + x^3 + x^2 + x + 1)$.
   (d) Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{F}_2$.
   (e) Deduce that $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Q}$.
   (f) Show that the splitting field of $x^4 + x^3 + x^2 + x + 1$ over $\mathbb{Q}$ has degree 4.
   (g) Show that the splitting field L of $x^5 - 2$ has degree 20.
   (h) Show that $\mathrm{Aut}_{\mathbb{Q}}(L)$ has a normal subgroup of order 5.
   (i) Show that $\mathrm{Aut}_{\mathbb{Q}}(L)$ is not Abelian. (Hint: Show that is has a non-normal subgroup)

2. (a) Define what it means for a group to be solvable.
   (b) Give an example of a non-solvable group. You do not have to prove that it is not solvable.
   (c) Let G be a solvable group. Show that the subgroup generated by $\{g^{-1}h^{-1}gh \mid g, h \in G\}$ is not equal to G.
   (d) Show that a group of order 28 is solvable. (Hint: Show that it has a normal subgroup of order 7, you may assume groups of order p^2 are Abelian.)

3. Let $f(x) = x^8 - 4x^4 + 2$.
   (a) Define what it means for an extension to be radical.
   (b) What is the relationship between solvable groups and radical extensions?
   (c) Show that $L = \mathbb{Q}(i, \sqrt[4]{2+\sqrt{2}}, \sqrt[4]{2-\sqrt{2}})$ is the splitting field of $f(x)$.
   (d) Show that L is a radical extension. Hint: you will have to write L as $\mathbb{Q}(a_1, a_2, a_3, a_4)$ for appropriately chosen a_i.

4. Let $L/\mathbb{F}_p$ be a field extension.

   (a) Define the degree of a field extension.

   (b) Show that if $[L : \mathbb{F}_p] = n$, then $|L| = p^n$.

   (c) Let $m \in \mathbb{N}$ Let $K = \{\alpha \in L \mid \alpha^{p^m} = \alpha\}$ is a subfield of L.

   (d) Let $f(x) = x^4 - 1 \in \mathbb{F}_3[x]$. Let L be the splitting field of $f(x)$. Find the degree $[L : \mathbb{F}_3]$.

   (e) What is the Galois group $\mathrm{Aut}_{\mathbb{F}_3}(L)$?

   (f) Construct a finite field of order 9. Write down the inverse of each element.