Math 146 Problem Sheet 1: Polynomials, Symmetric Functions and PIDs

Robert Kropholler

January 29, 2019

1. The purpose of this exercise is to show you that while Cardano's formula gives you the roots of any cubic polynomial, it does not necessarily give them to you in the simplest possible form. This was a cause of much consternation among Renaissance-era mathematicians, and a major impetus to thinking more carefully about complex numbers.

   (a) Consider the polynomial $f(x) = x^3 + x - 2$. Find the three roots of this polynomial without using Cardano's formula.

   (b) Now write down the three roots of this polynomial using Cardano's formula.

   (c) Use your answers to (a) and (b) to deduce the surprising formula:

   $$1 = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}}$$

2. Write the following symmetric functions as polynomials in the elementary symmetric functions $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{Z}[x_1, x_2, x_3]$.

   (a) $x_1^2 + x_2^2 + x_3^2$

   (b) $x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2$

3. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of the polynomial $f(x) = x^3 - 7x^2 + 2x + 1$.

   Write down an explicit cubic polynomial whose roots are $\alpha_1^2, \alpha_2^2, \alpha_3^2$. You may find your solutions to the previous question useful.

4. Suppose $\alpha, \beta, \gamma \in \mathbb{C}$ satisfy the following equations

   $$\begin{aligned} \alpha + \beta + \gamma &= 3 \\ \alpha^2 + \beta^2 + \gamma^2 &= 5 \\ \alpha^3 + \beta^3 + \gamma^3 &= 12 \end{aligned}$$

   Prove that $\alpha^n + \beta^n + \gamma^n \in \mathbb{Z}$ for all $n \geqslant 4$.

5. Show that $\mathbb{Z}[x]$ is not a principal ideal domain. (Hint: consider the ideal generated by 2 and x.)