Sheet 2: Irreducibility and field extensions

Robert Kropholler

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Let $\mathbb{F}_p$ be the finite field with p elements.

1. (a) Prove the rational roots theorem: Let $f(x) = a_n x^n + \ldots + a_0 \in \mathbb{Z}[x]$, and let $p/q \in \mathbb{Q}$ be a rational number in reduced form, i.e. such that $gcd(p, q) = 1$. If p/q is a root of f, then $p|a_0$ and $q|a_n$.
   (b) Use (a) to prove that $x^3 + 3x + 3$ is irreducible in $\mathbb{Q}[x]$, and conclude that $\mathbb{Q}[x]/(x^3 + 3x + 3)$ is a field.
   (c) Let α denote the image of x in $\mathbb{Q}[x]/(x^3 + 3x + 3)$. Express each of $1/\alpha$, $1/(1+\alpha)$, $1/(1+\alpha^2)$ in the form $c_2\alpha^2+c_1\alpha+c_0$ with $c_0, c_2, c_2 \in \mathbb{Q}$.

2. (a) Write down the 4 possible degree 2 polynomials in $\mathbb{F}_2[x]$. Which are irreducibe?
   (b) Show that if $f(x)$ is a degree 4 polynomial which is reducible, then it either has a root or can be factored as a product of two irreducible quadratics.
   (c) Show that $x^4 + x + 1$ is irreducible in $\mathbb{F}_2$.
   (d) Show that $5x^4 + 3x + 7$ is irreducible in $\mathbb{Q}[x]$.

3. Show that the following are irreducible in $\mathbb{Q}[x]$.
   (a) $2x^4 + 15x^2 + 10$.
   (b) $x^6 + x^3 + 1$.
   (c) $x^4 + 4x^3 + 6x^2 + 6x + 5$.

4. Let L/K be a field extension where $[L : K]$ is prime.
   (a) Show that there are no fields M such that $K \subset M \subset L$.
   (b) Show that L/K is a simple extension.

5. The formal derivative $D\colon K[x] \to K[x]$ is defined by $D(a_0 + a_1x + \cdots + a_nx^n) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$. Prove that if $a, b \in K$ and $f, g \in K[x]$ then
   (a) $D(af + bg) = aDf + bDg$;

(b) $D(fg) = fDg + gDf$;

(c) $Dh(x) = Dg(x)Df(g(x))$ when $h(x) = f(g(x))$. If $a \in K$ show that

(d) $(x - a)$ divides $f(x)$ in $K[x]$ if and only if $f(a) = 0$;

(e) $(x - a)^2$ divides $f(x)$ in $K[x]$ if and only if $f(a) = 0 = Df(a)$.

Deduce that if the polynomials f and Df have no common factors in $K[x]$, then f has no multiple root.