Sheet 3: Splitting fields, normal and separable extensions.

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February 22, 2019

1. Show that if L/K is a finite degree extension, then every element of L is algebraic over K.
2. Let α, β be algebraic over $\mathbb{Q}$. Show that $\alpha - \beta, \alpha\beta, \alpha^{-1}$ are all algebraic over $\mathbb{Q}$.
3. Let $\bar{\mathbb{Q}} = \{\alpha \mid \alpha \text{ is algebraic over } \mathbb{Q}\}$, this is the *algebraic closure of* $\mathbb{Q}$. Using the above or otherwise show that $\bar{\mathbb{Q}}$ is a field.
4. Suppose that K is a finite extension of $\mathbb{Q}$ and that α is algebraic over K. Use the tower law to show that α is algebraic over $\mathbb{Q}$.
5. We say that a field L is *algebraically closed* if given any polynomial with coefficients in L has a root in L. Deduce that $\bar{\mathbb{Q}}$ is algebraically closed.
6. It is unknown whether $e + \pi$ or $e\pi$ are transcendental numbers. Prove that at least one of them is.
7. Show that $\bar{\mathbb{Q}}$ is not a finite extension of $\mathbb{Q}$.
8. By considering the polynomial $ax^2 - b$ or otherwise.
   (a) Show that if $b \neq n^2$ for any integer n. Then $\sqrt{b}$ is irrational.
   (b) Show that if a and b are coprime integers and $a, b \neq n^2$ for any integer n, then $\sqrt{a} \notin \mathbb{Q}(\sqrt{b})$.
9. Find the splitting field and the degree over $\mathbb{Q}$ for each of the following:
   (a) $(x^2 - 2)(x^2 - 5)$
   (b) $x^3 - 2$
   (c) $x^4 - 2$
10. Let M/L, L/K be field extensions. Find a counter example for each of the following.
    (a) If M/L and M/K are normal, then L/K normal.
    (b) If L/K is normal, then M/L normal.
    (c) If L/K is normal, then M/K normal.