# Sheet 4

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1. Let $L/\mathbb{Q}$ be a field extension. Let $\phi \in \mathrm{Aut}_{\mathbb{Q}}(L)$. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial. Show that $f(\phi(\alpha)) = \phi(f(\alpha))$ for all $\alpha \in L$.

2. Let $L/\mathbb{Q}$ be a finite degree field extension. Show that complex conjugation defines an element of $\mathrm{Aut}_{\mathbb{Q}}(L)$.

3. Show that an extension $L/\mathbb{Q}$ of degree 2 is always normal. What is the Galois group of such an extension?

4. Let $L$ be the splitting field of the polynomial $x^8 - 1$ over $\mathbb{Q}$ show the following:
   (a) $L = \mathbb{Q}(i, \sqrt{2})$
   (b) $[L : \mathbb{Q}] = 4$
   (c) $\mathrm{Aut}_{\mathbb{Q}}(L) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

5. Let $L$ be the splitting field of $x^4 + 1$ show that $\mathrm{Aut}_{\mathbb{Q}}(L) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

6. (a) Show that $\sqrt{2+\sqrt{2}}$ satisfies the polynomial $x^4 - 4x^2 + 2$.
   (b) Show that the four roots of this polynomial are $\pm\sqrt{2 \pm \sqrt{2}}$.
   (c) Show that $\sqrt{2} \in \mathbb{Q}(\sqrt{2+\sqrt{2}})$.
   (d) Show that $(\sqrt{2+\sqrt{2}})(\sqrt{2-\sqrt{2}}) = \sqrt{2}$.
   (e) Deduce that the splitting field of $x^4 - 4x^2 + 2$ is $\mathbb{Q}(\sqrt{2+\sqrt{2}})$
   (f) Let $\phi \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{2+\sqrt{2}}))$ be the automorphism such that $\phi(\sqrt{2+\sqrt{2}}) = \sqrt{2-\sqrt{2}}$. Show that $\phi^2$ is not the identity.