Sheet 7: Construction and Solvable extensions

Robert Kropholler

April 17, 2019

Useful throughout will be the fact that if N is a normal subgroup of G and $n \in N, g \in G$ then $gn = n'g$ for some $n' \in N$.

1. Recall the second isomorphism theorem

   Theorem 0.1. *Let G be a group and H, N be subgroups. Suppose that N is a normal subgroup of G. Then $HN/N \cong H/N \cap H$.*

   (a) Let G be a group and H be a subgroup of G. Suppose that K is a normal subgroup of G. Show that $K \cap H$ is a normal subgroup of H.

   (b) Suppose that N, M are subgroups of G with $M \lhd N$. Let $N_H = N \cap H$ and $M_H = M \cap H$. Show that $N_H M$ is a subgroup of N.

   (c) Show that $N_H M/M$ is a subgroup of N/M.
   Assume now that N/M is Abelian.

   (d) Deduce that $N_H M/M$ is Abelian.

   (e) Use the second isomorphism theorem to show that $N_H \big/ M_H$ is Abelian.

   (f) Show that if G is a solvable group and H is a subgroup, then H is a solvable group.

2. Recall the third isomorphism theorem

   Theorem 0.2. *Let G be a group and H, N be subgroups. Suppose that $N \leqslant H$ and $H, N \lhd G$. Then $G/N \big/ H/N \cong G/H$.*

   (a) Let G be a group and N be a normal subgroup. Suppose that $H_2 \lhd H_1 \leqslant G$. Let $H_i N = \{hn \mid h \in H_i, n \in N\}$. Show that $H_2 N \lhd H_1 N$.

   (b) Let $\phi\colon H_1 \to H_1 N \big/ H_2 N$ be given by $\phi(h) = hH_2N$. Show that the kernel contains H_2.

   (c) Show that ϕ is surjective. Hint: Use normality to show that $hnH_2N = hH_2N$.

   (d) Suppose that $H_1 \big/ H_2$ is Abelian. Show that $H_1 N \big/ H_2 N$ is Abelian.
   Hint: Show that it is a quotient of $H_1 \big/ H_2$.

(e) Use this to show that ${}^{G}\!/_{N}$ is solvable if G is solvable. Hint: The subgroups look like ${}^{H_iN}\!/_{N}$.

3. Let $V = \{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \leqslant A_4 \leqslant S_4$. Show that $V \lhd A_4$ and that ${}^{A_4}\!/_{V}$ is abelian. Deduce that S_4 is solvable. Deduce that there is a formula for the quartic.

4. (a) Let P be the regular n sided polygon with a vertex at $(1, 0)$. Show that the other vertices of P correspond to roots of unity.

   (b) Deduce that P is constructible if and only if all the n-th roots of unity are constructible.

   (c) Show that equilateral triangles are constructible and polygons with 3×2^m sides are constructible. Hint: Bisection of angles is possible.

   (d) Show that a nine-sided polygon is not constructible. Hint: Look at the proof that angle trisection is not possible.

5. Look at the following paper on origami `https://divisbyzero.com/2012/06/01/angle-trisection-using-origami/`. Fold an angle into a piece of paper with one fold. Trisect this angle using origami and hand in this trisection.