Sheet 8: Sylow Theory

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April 26, 2019

1. (a) Show that A_4 has 1 or 3 Sylow 2-subgroup and 1 or 4 Sylow 3-subgroups.
   (b) Write down the unique Sylow 2-subgroup.
   (c) Show that the Sylow 2-subgroup is Abelian.
   (d) Show that if there were one Sylow 3-subgroup, then A_4 would be Abelian.
   (e) Deduce that there are 4 Sylow 3-subgroups.
   (f) Write down the 4 Sylow 3-subgroups.

2. Show that a group of order 100 has a normal subgroup of order 25.

3. Show that a group of order $20449 = 11^2 13^2$ has a normal subgroup of order 169.

4. (a) Let G be a group and H, K be subgroups of order p, where p is prime. Show that $H \cap K = \{e\}$ or $H = K$. Hint: What is the order of an element in $H \cap K$.
   (b) Show that if $H \cap K = \{e\}$, then G has at least $2(p-1)$ elements of order p.

5. Let G be a group of order 56.
   (a) Show that $n_2 = 1$ or 7 and $n_7 = 1$ or 8.
   (b) Show that any Sylow 2-subgroup has order 8.
   (c) Suppose that $n_7 = 8$ show that G has 48 elements of order 7.
   (d) Show that if $n_7 = 8$, then there $n_2 = 1$ (Hint: Consider how many other elements of G are not in Sylow 7-subgroups.)
   (e) Deduce that G has a normal subgroup of order 7 or order 8.

6. Let $p < q < r$ be distinct primes. Let G be a group of order pqr.
   (a) Show that $n_r = 1$ or pq.
   (b) Suppose $n_r = pq$, show that there are $pq(r-1)$ elements of order r.

(c) Show that $n_q = 1$ or $n_q \geqslant r$ and $n_p = 1$ or $n_p \geqslant q$.

(d) Suppose that $n_q \geqslant r$, show that there are at least $r(q-1)$ elements of order q.

(e) Suppose $n_p \geqslant q$, show that there are at least $q(p-1)$ elements of order p.

(f) Show that if $n_p, n_q, n_r > 1$, then there are at least $pq(r-1) + q(p-1) + r(q-1) = pqr + qr - q - r$ elements of G.

(g) Since $pqr + qr - q - r \leqslant pqr$, show that $q \leqslant \frac{r}{r-1}$.

(h) Show that $q > 2$ and that $\frac{r}{r-1} \leqslant 2$.

(i) Deduce that at least one of n_p, n_q, n_r are 1.

(j) Bonus: Use this to show that G is solvable.